Picking A Thorny Rose: Optimal Timing of Liquidity Premium*

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Keywords: portfolio choice; transaction cost; bid-ask spread; liquidity premium.

JEL Classification: G11; C61.

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Abstract

Stock illiquidity varies over time and predicts future returns, giving investors an option to time the liquidity premium. Capturing the liquidity premium requires proper management of the liquidity cost. We study a dynamic trading model with stochastic bid-ask spread and the return predictability derived from it. The optimal trading strategy exhibits three distinct features: it captures the spot liquidity premium by purchasing at large spreads, it avoids over-trading by choosing wide inactive regions, and it aims in front of the target to chase the future liquidity premium. We show that a strong interaction of trading costs and return predictability leads to substantial losses from suboptimal trading and that timing the liquidity premium optimally allows the investor to benefit from a higher illiquidity risk. These results continue to hold when the investor is averse to poor performance against a liquid benchmark.

1 Introduction

Stock illiquidity is a double-edged sword for investors. Indeed, it either impedes investors from quickly acting on their information or results in transaction costs that lower their net returns. In contrast, stock illiquidity is time-varying and predicts future returns (e.g. Chen et al. (2018)), which implies the possibility of trading strategically to capture the liquidity premium.[1] Any attempt to trade on the liquidity premium inevitably entails a tradeoff—earning a high liquidity premium is likely associated with high liquidity costs. Addressing this tradeoff is important in successfully timing the liquidity premium.

Since the seminal work of Constantinides (1986), many studies have examined optimal portfolio choice with stock illiquidity.[2] Studies in this area find that even small transaction costs can make investors' optimal trading strategies deviate considerably from no-transaction cost benchmarks such as those implied by Merton (1969) and Merton (1971). Such deviation reflects a tradeoff between the gains from an improvement in portfolio composition and the losses from trading costs. However, no study has simultaneously incorporated randomness and the predictive power of stock illiquidity.

This study contributes to the literature by solving a dynamic trading model with stochastic stock bid-ask spread (BAS hereafter) and the return predictability derived from it. In the model, an investor trades in three assets: a liquid risk-free bond, a liquid stock, and an illiquid stock. Trading in the liquid bond and liquid stock is costless, while trading in the illiquid stock requires exceeding its BAS. The investor's objective is to maximize their expected utility from their net wealth in a finite horizon. Thus, our model serves as a natural extension of the one proposed in Dai et al. (2011) (DJL model hereafter).[3] To the best

[1]Liquidity premium refers to the extra expected returns on illiquid stocks that compensate for the liquidity costs incurred due to holding or trading in these stocks. Amihud and Mendelson (1986), Datar et al. (1998), Amihud (2002), Pástor and Stambaugh (2003), and Chen et al. (2018) document the presence of liquidity premium in common stocks. Brenner et al. (2001) and Christoffersen et al. (2018) document that stock options have a considerable liquidity premium. Although Ben-Rephael et al. (2015) argue that stock market liquidity has improved, illiquid stocks continue to have a considerable spread (e.g., Abdi and Ranaldo (2017)).

[2]See, for example, Davis and Norman (1990), Longstaff (2001), Liu and Loewenstein (2002), Liu (2004), Lo et al. (2004), Lynch and Tan (2010), Gârleanu and Pedersen (2013), and Ang et al. (2014).

[3]To separate the effect of the randomness of the BAS and its predictive power, we also consider an intermediate model in which illiquid stock has a stochastic BAS but constant expected returns. For ease of reference, we call this model the stochastic bid-ask spread (SBAS)-DJL model. Comparing our model with

of our knowledge, this is the first portfolio choice model that simultaneously incorporates randomness and the predictive power of the BAS.

We use a portfolio consisting of S&P 500 stocks as a proxy for liquid stock and a portfolio consisting of small cap stocks as a proxy for illiquid stock. The spreads of small cap stocks are estimated at the stock level following Abdi and Ranaldo (2017).[4] By estimating a predictive regression following Piotroski and So (2012), we first confirm that the spreads indeed have predictive power on the future returns of these small cap stocks.[5] Next, we take the cross-stock average of stock-level spreads to obtain the portfolio-level spread. Afterward, we use the quasi-maximum likelihood estimation (QMLE) method to estimate the model parameters. Our parameter estimates suggest a strong predictive power of the BAS on future returns, as a 1% increase in portfolio-level BAS is associated with an approximately 7.7% increase in the annualized expected return of the portfolio.

Next, we perform a numerical analysis to uncover the model's implications. In our model, the set of state variables that characterize the investor's portfolio allocation includes the current BAS and the share of their wealth invested in the illiquid stock, and the investor's optimal trading strategy is characterized by the sell and buy boundaries that jointly delimit a no-trade region. When the state variables lie above (or below) the sell (or buy) boundary, it is optimal to sell (or buy) the minimum amount of illiquid stock so that the after-trade allocation lies exactly on the sell (or buy) boundary; when the state variables lie in the no-trade region, it is optimal not to trade in the illiquid stock, as the trading costs would exceed the benefits of a portfolio rebalancing.

Unlike other transaction cost models, our model features a strong interaction of return predictability and trading costs, as they are both directly related to the BAS. This unique feature leads to some interesting patterns in the optimal trading strategy, which we illustrate as follows:

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the SBAS-DJL model allows us to understand the effect of the predictive power of the BAS, and comparing the SBAS-DJL model with the DJL model allows us to uncover the effect of the randomness of the BAS alone.

[4]Abdi and Ranaldo (2017) show that the estimated spreads are small for large cap stocks.

[5] We focus on the predictive power of effective spread constructed following Abdi and Ranaldo (2017). Chen et al. (2021) show that the effective trading costs constructed following Hasbrouck (2009) also predict future stock returns.

- **Capture Spot Liquidity Premium** In our model, because the BAS positively predicts the future returns of the illiquid stock, the levels of both the buy and sell boundaries increase with an increase in the BAS. Meanwhile, the no-trade region widens as the BAS increases because a higher BAS implies higher trading costs. Therefore, the investor desires greater exposure to the illiquid stock as its BAS widens, but they also perform less frequent portfolio rebalancing to save on trading costs. Through Monte Carlo simulations, we show that the investor purchases a substantial amount of illiquid stock at a large spread, which confirms that capturing the spot liquidity premium is indeed an important component of the optimal trading strategy.

- **Control Turnover** Implementing such a premium-timing strategy is inherently costly as capturing a high spot liquidity premium requires the investor to trade at a large BAS. The investor must carefully control turnover to ensure that the trading costs do not erode the profitability of such a strategy. This intuition is confirmed by the fact that the optimal no-trade region in our model is much wider than that in the DJL model, which does not allow the investor to time the liquidity premium. The simulation results indicate that our model continues to imply a higher turnover of the illiquid stock than the DJL model does. However, the extra trading costs are well compensated for by the higher returns from timing the liquidity premium.

- **Aim in Front of the Target** Consistent with the empirical pattern, we assume that the BAS of the illiquid stock follows a mean-reverting process. When the BAS is well below the mean, it tends to increase in the near future. In this case, it is optimal for the investor to buy some illiquid stock in advance because this stock's expected return is likely to increase in the near future as the BAS rebounds and buying shares at a narrow spread implies lower trading costs. Similarly, when the BAS is well above the mean, it tends to decrease in the near future. In this case, mean reversion should weaken the investor's incentive to purchase shares at a large BAS. Thus, our model implies an aiming-in-front-of-target trading pattern.[6] This intuition is formally verified

[6]The aim-in-front-of-the-target pattern is present in the models ofGârleanu and Pedersen (2013) and Dai et al. (2022). Nonetheless, the mechanism of the two models is different from that in our model. In Gârleanu

in a comparison of the optimal strategy and a myopic strategy that ignores the future variations in the BAS.

Next, we show that adopting the optimal trading strategy in the context of timing the liquidity premium is particularly important because of the strong interaction of return predictability and trading costs. To make this point, we first calculate the investor's utility loss incurred by adopting the myopic strategy instead of the optimal one.[7] When adopting the myopic strategy, although the investor still trades in response to changes in the liquidity premium, they overtrade and consequently incur high trading costs. Hence, the myopic strategy causes a large economic utility loss—in the base case, with no portfolio constraints, it is as large as 12.2% (2.44% annually) of the investor's initial level of wealth; if we constrain the investor from borrowing and short-selling, the loss remains at 3.3% (0.66% annually) of their initial level of wealth. In comparison, if we eliminated the interaction of return predictability and trading costs by waiving trading costs, this utility loss would become much smaller, indicating that such interaction is the mechanism behind the results.[8]

The studies on portfolio choice with BAS often ignore the predictive power of the BAS on future stock returns. In our model, if the investor erroneously assumes a constant expected return on the illiquid stock, then their portfolio choice strategy would not respond to the changes in the stock's expected return. We show that the resultant utility loss can be substantial, especially for long-term investors. For example, with an investment horizon of 10 years, an unconstrained investor would incur an equivalent wealth loss of 0.96% annually by ignoring the return predictability derived from the illiquid stock's BAS; if the investor were subject to strict no-leverage and no-short-sale constraints, then the loss would be lower at 0.45% annually because of the reduced investment in the illiquid stock.

Furthermore, we use historical market return data to retrospectively test the trading s-

and Pedersen (2013) and Dai et al. (2022), trading in the illiquid asset is subject to quadratic transaction costs. Hence, it is optimal for the investor to trade slowly toward the target to save on transaction costs, which requires the investor to aim in front of the target. In our model, trading costs are linear, and the mean reverting attribute of the BAS drives the aiming-in-front-of-target pattern.

[7]The myopic strategy is constructed under the assumption that the investor responds to the spot BAS in every moment and ignores the future variations in the BAS.

[8]To do this, we examine alternative portfolio choice models in which there is no transaction cost, no return predictability, or a constant rate of transaction cost.

trategies, and we find that the optimal trading strategy generates a much higher investment return and Sharpe ratio than the myopic strategy and the no-liquidity-premium-timing s-trategy do. For an unconstrained investor, adopting the optimal trading strategy generates an average monthly return of 1.70% and a Sharpe ratio of 30.71%. In comparison, adopting the myopic trading strategy (or no-liquidity-premium-timing strategy) generates an average monthly return of 1.33% (or 1.07%) and a Sharpe ratio of 25.8% (or 29.71%). We find qualitatively similar results when the investor is restrained from borrowing and short selling. A subsample analysis suggests that the performance advantage of optimal liquidity premium timing is concentrated during adverse events, such as the 2008–09 global financial crisis.

As we stated earlier, stock illiquidity can have both positive and negative effects on investors' trading performance, and its net effect depends on which effect is dominant. We show that a higher illiquidity risk, in the form of higher BAS volatility, can benefit the investor under optimal liquidity premium timing. In contrast, if the investor fails to time the liquidity premium, then the higher BAS volatility makes them slightly worse off. Hence, we conclude that optimal liquidity premium timing is crucial for the investor to benefit from greater variability in stock market liquidity.

We further study an optimal liquidity premium timing problem faced by a mutual fund manager who exhibits aversion to poor performance against a liquid benchmark. This analysis is motivated by the following three observations. First, in the U.S., many equity mutual fund managers use portfolios of liquid stocks, such as S&P 500 stocks, as their performance benchmarks and the performance of the managed portfolios against the benchmarks is closely related to managers' reputation, status, and compensation (e.g., Ma et al. (2019)). Second, holding illiquid assets to extract the liquidity premium appears to be a popular strategy in the mutual fund industry (e.g., Huang et al. (2011)). Third, for fund managers with liquid benchmarks, adopting a liquidity premium-timing strategy may drive the composition of the managed portfolios away from the benchmark and can be quite risky. Therefore, we expect the presence of relative performance concerns to affect managers' optimal liquidity premium timing strategies.

We show that because of managers' aversion to poor performance, they allocate less wealth to the illiquid stock and time the liquidity premium less aggressively. Nonetheless, the utility losses from adopting suboptimal trading strategies remain economically significant. Moreover, we find that a moderate degree of relative performance concerns increases the magnitude of utility gains from performing optimal liquidity premium timing. Last, we show that when the fund manager times the optimal liquidity premium, the greater variability in the BAS makes both the fund manager and fund investor better off.[9]

Other Related Studies In our model, we take the pricing effect of stock illiquidity (which is empirically found to be strong) as exogenously given. Note that this empirical magnitude is not entirely in line with the traditional portfolio theories. For example, Amihud (2002) documents a liquidity premium-to-transaction cost (LPTC) ratio of 1.9 for the NYSE stocks, while that of Constantinides (1986) is below 0.1. To explain the magnitude of the LPTC ratio, studies have extended the model of Constantinides (1986) in various directions. The literature suggests that a high LPTC ratio is generated by models with a time-varying investment opportunity set (Jang et al. (2007), Lynch and Tan (2011), Dai et al. (2016)), incomplete information on shifts in market regime (Chen et al. (2021)), or convexity in investors' preferences (Dai et al. (2021)).

Our study is also related to the studies on portfolio choice with return predictability. It is empirically found that some economic variables, including dividend yield (e.g., Campbell and Shiller (1988)), consumption-to-wealth ratio (e.g., Lettau and Ludvigson (2001)), real interest rate (e.g., Campbell (1987)), and investor sentiment (e.g., Huang et al. (2015) and Jiang et al. (2019)), can predict future stock returns.[10] The theoretical studies on portfolio choice with predictable returns include Balduzzi and Lynch (1999), Campbell and Viceira (1999), Barberis (2000), Xia (2001), Huang and Liu (2007), Wachter and Warusawitharana (2009), Lynch and Tan (2010), Gârleanu and Pedersen (2013), and Michaelides and Zhang (2022). Recently, Moreira and Muir (2017) and Moreira and Muir (2019) demonstrate the economic importance of volatility timing. These studies often assume that the predictive

[9]The fund investor delegates the management of their wealth to the fund manager and exhibits no preoccupation with performance. When calculating the fund investor's utility, we use the fund manager's optimal trading strategy as the input.

[10]It has been found that some derivative-based measures also exhibit predictive power on future stock returns (e.g., Cremers and Weinbaum (2010), An et al. (2014), and Ge et al. (2016)). Accordingly, using options-implied measures to improve the portfolio choice has been proposed (e.g. DeMiguel et al. (2014)).

signal is not directly linked to the stock's trading cost. However, in our model, the predictor itself (i.e., the BAS) is strongly related to trading cost, which implies that the investor must make a better trade-off between the profits and costs involved in active timing.

The remainder of this paper is organized as follows. In the next section, we present the model setting and the motivation behind it. In Section 3, we perform a numerical analysis of the model implications. In Section 4, we conclude the paper. Technical details are provided in the Appendix.

2 The Model

In this section, we present the baseline theoretical framework adopted in this study.

2.1 Motivation

There are numerous studies on optimal portfolio choice with proportional transaction costs. Most of them assume a constant rate of transaction costs or the costs incurred in a trade to be a constant fraction of the dollar trading volume. Here, we perform an intuitive analysis to demonstrate that in reality, the effective BAS of illiquid stocks is time-varying and has predictive power on future stock returns. To the best of our knowledge, no optimal portfolio choice model has simultaneously incorporated these two stylized facts of the BAS.

Chen et al. (2018) document the predictive power of aggregate market illiquidity on aggregate market returns. To ensure consistency with our theoretical framework, our analysis in this subsection exclusively focuses on the predictive power of the BAS of illiquid stocks.

2.1.1 Sample of Illiquid Stocks and Calculation of the BAS

Our data are obtained from the Center for Research in Security Prices. We use data on stocks with small market capitalization (i.e., small cap stocks) because they are generally less liquid than large cap stocks. We choose a sample period from January 2002 to December 2021, as this allows us to avoid the mechanical effect of decimalization, which occurred in 2001, on stock spreads.

We select sample stocks according to the following two criteria. The stocks should be in the lowest decile of market capitalization as of January 2002 and have complete price information—daily low/high and closing prices—over the sample period.[11] Applying this filter leaves us with 388 common stocks. For each of these stocks and for each month t, we use daily price information to estimate its effective BAS using the method proposed by Abdi and Ranaldo (2017).[12]

2.1.2 Predictive Power of the BAS on Stock Returns

Following Piotroski and So (2012), we verify that the BAS exhibits predictive power on the future returns in our sample of small cap stocks. First, we estimate the following predictive regression:

$$R_{i,t} = \beta_{0,t} + \beta_{1,t}\theta_{i,t-1} + \epsilon_{i,t} \tag{2}$$

for each month t, where $R_{i,t}$ is stock i's return in month t (calculated using its bid price) and $\theta_{i,t-1}$ is the estimated BAS of stock i for month $t-1$. Next, we take the time-series average of the parameter estimates. As shown in Table 1, the average of estimates $\hat{\beta}_{0,t}$ is $\hat{\beta}_0 = 0.0096$ with a Newey–West adjusted t-statistic of 2.63 and the average of estimates $\hat{\beta}_{1,t}$ is $\hat{\beta}_1 = 0.3702$ with a Newey–West adjusted t-statistic of 2.42. These results demonstrate the predictive power of the BAS in our sample.

Figure 1 shows the average BAS of the sample stocks with monthly frequency to demonstrate the time-variability of the spreads. We find that the average BAS indeed exhibits

[11] Price information is necessary to estimate the effective spreads of these stocks. Although the market capitalization of the sample illiquid stocks varies over time, they are a representative sample of illiquid stocks over our sample period. For example, in December 2021, 53.1% of the sample stocks were in the lowest decile of market capitalization and 86.1% were below the median market capitalization.

[12] The formula used for calculating the effective spread in month t is as follows:

$$\theta_t = \sqrt{\max\left\{0, \frac{4}{N}\sum_{i=1}^{N}(c_i - \eta_i)(c_i - \eta_{i+1})\right\}}, \tag{1}$$

where N is the number of trading days in month t, η_i is the average of the high and low prices on day i, and c_i is the closing price on day i. There are alternative methods for estimating the stock-level BAS in the literature (e.g., Roll (1984) and Corwin and Schultz (2012)). As a robustness check, we also use the methods in Roll (1984) and find similar results. These findings are consistent with the argument of Abdi and Ranaldo (2017) that their BAS estimate is highly correlated with existing ones .

Table 1: Coefficient Estimates

This table summarizes the coefficient estimates of the predictive regression $R_{i,t} = \beta_{0,t} + \beta_{1,t}\theta_{i,t-1} + \epsilon_{i,t}$, where $R_{i,t}$ is stock i's return in month t (calculated using its bid price) and $\theta_{i,t-1}$ is the estimated BAS of stock i for month $t-1$. $\hat{\beta}_0$ (or $\hat{\beta}_1$) is the time-series average of estimate $\hat{\beta}_{0,t}$ (or $\hat{\beta}_{1,t}$).

Coefficient	Estimate	Newey–West Adjusted t-Statistic
$\hat{\beta}_0$	0.0096	2.63
$\hat{\beta}_1$	0.3702	2.42

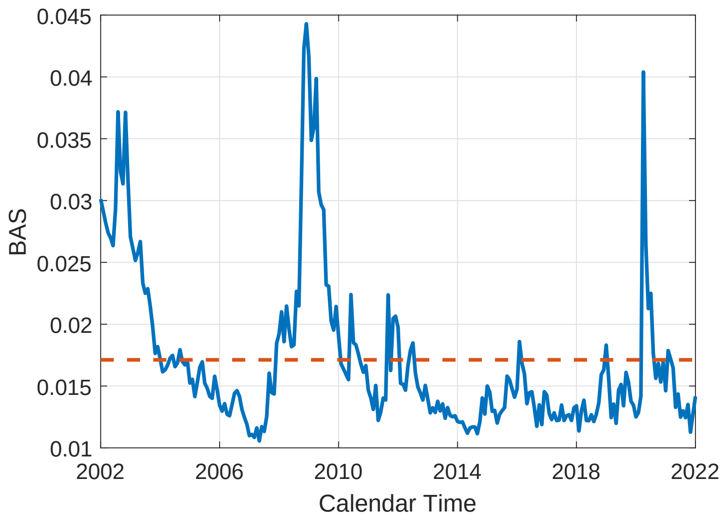


Figure 1: Time Series of Bid-Ask Spread

This figure shows the monthly estimates of the BAS for the portfolio of small cap stocks used in this study. The stock-level BAS is constructed following Abdi and Ranaldo (2017) and the portfolio-level BAS is the equal-weighted average of the stock-level BAS. The red dashed line is the time-series mean of the BAS.

significant variations over time, with a mean value of 1.7%.

2.2 Model Setting

Next, we present the baseline model, which is based on the above analysis.

2.2.1 Assets

The investment opportunity set in our model includes three assets. The first is a risk-free bond that offers interest at a constant rate r. The second is liquid stock whose price process

S_{Lt} follows a geometric Brownian motion as follows:

$$\frac{dS_{Lt}}{S_{Lt}} = \mu_L dt + \sigma_L dZ_{1t}. \tag{3}$$

Trading in the bond and liquid stock is costless. The third asset is illiquid stock, wherein trading requires the investor to exceed its BAS. Based on the empirical evidence, we assume that the bid price process S_{It} of the illiquid stock is governed by the following stochastic differential equation.

$$\frac{dS_{It}}{S_{It}} = (\mu_I + \lambda\theta_t)dt + \sigma_I dZ_{2t}, \tag{4}$$

where $\theta_t \geq 0$ is the time-t BAS and $\lambda \geq 0$ is the LPTC ratio (e.g., Constantinides (1986)). Thus, the stock's ask price at which the investor can purchase shares is $(1+\theta_t)S_t$. We assume that θ_t follows a square root process to ensure the non-negativity of the BAS.

$$d\theta_t = \kappa(\eta - \theta_t)dt + \nu\sqrt{\theta_t}dZ_{3t}, \tag{5}$$

where $\eta > 0$ is the long-term mean value, $\kappa > 0$ is the mean reversion speed, and ν is the volatility of the BAS.[13]

In equations (3), (4), and (5), Z_{1t}, Z_{2t} and Z_{3t} are three standard Brownian motion processes, defined on a filtered complete probability space $(\Omega, \mathcal{F}_t, P)$, with the following correlation coefficient matrix.

$$\Lambda = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}. \tag{6}$$

Usually, in portfolio choice problems, the liquid (or illiquid) stock in the model is interpreted as a diversified portfolio consisting of liquid (or illiquid) stocks. Our interpretation is consistent with this calibration strategy (see Section 3.1).

[13] Note that the process θ_t governed by the stochastic differential equation (5) is unbounded from above. If we assume that the illiquid stock's middle price follows the process (4) instead of its bid price, then it is possible to obtain a negative bid price when the spread θ_t is sufficiently large, which is unreasonable. Hence, we model the illiquid stock's bid price using (4). One alternative modeling approach is to set a reflecting boundary at zero bid price and model the middle price. The implications of such a model will be similar to those derived from our model.

2.2.2 Budget Constraints and Objective Function

When there are transaction costs on the illiquid stock, we need to track the investor's liquid and illiquid holdings separately. Let x_t be the dollar value invested in the liquid assets with a dollar amount of ω_t invested in the liquid stock, y_t be the dollar value invested in the illiquid stock (measured by its bid price), and D_t and I_t be two non-decreasing adaptive processes denoting the cumulative selling and purchasing dollar values of the illiquid stock (also measured by its bid price). Then, the investor's dynamic budget constraints are as follows:

$$dx_t = (rx_t + \omega_t(\mu_L - r))dt + \omega_t\sigma_L dZ_{1t} + dD_t - (1+\theta_t)dI_t, \tag{7}$$

$$dy_t = (\mu_I + \lambda\theta_t)y_t dt + \sigma_I y_t dZ_{2t} - dD_t + dI_t. \tag{8}$$

The investor's objective is to maximize their expected utility from the level of net wealth at a finite horizon T by choosing the optimal trading strategy for both the liquid and illiquid stock:

$$\max_{\{(D_t, I_t, \omega_t): t \geq 0\}} E\left[\frac{1}{1-\gamma} W_T^{1-\gamma}\right] \tag{9}$$

subject to (5), (7), and (8), where $W_t = x_t + y_t^+ - (1+\theta_t)y_t^-$ is the net wealth process and $\gamma > 0$ is the constant relative risk aversion coefficient of the investor. For any admissible trading strategy, we require the resultant wealth process to satisfy $W_t \geq 0$ for all t.

Portfolio Constraints In practice, investors' portfolio choices are subject to certain constraints. For ease of exposition, we focus on two cases—the case without any position limits (the Unconstrained Case" hereafter) and the case with strict no-leverage and no-short-sale constraints on both stocks (the Constrained Case" hereafter). In the Constrained Case, we require the investor's subwealth processes to satisfy $0 \leq y_t \leq W_t$ and $0 \leq \omega_t \leq x_t$ for all $t \geq 0$.[14]

[14] Note that incorporating other types of portfolio constraints in our model is straightforward. Our analysis also suggests that the results are likely to hold in other intermediate cases.

2.2.3 HamiltonJacobi–Bellman Equation

The investor's optimization problem is solved using the dynamic programming method. Thus, we define the investor's value function as follows:

$$J(x, y, \theta, t) = \max_{\{(D_s, I_s, \omega_s): s \geq t\}} E\left[\frac{1}{1-\gamma} W_T^{1-\gamma} | (x_t, y_t, \theta_t) = (x, y, \theta)\right]. \tag{10}$$

Shreve and Soner (1994) suggests that the function $J(x, y, \theta, t)$ is the viscosity solution to the following HamiltonJacobi–Bellman (HJB) equation.

$$\max\Big\{\sup_{\omega} \mathcal{L}_0^{\omega} J + J_t,\ \ \mathcal{S}_0 J,\ \ \mathcal{B}_0 J\Big\} = 0, \tag{11}$$

$$J(x, y, \theta, T) = \frac{1}{1-\gamma}(x + y^+ - (1+\theta)y^-)^{1-\gamma}, \tag{12}$$

on an appropriate solution domain,[15], where the differential operators in (11) are given by

$$\begin{aligned} \mathcal{L}_0^{\omega} J &= (rx + \omega(\mu_L - r))J_x + (\mu_I + \lambda\theta)yJ_y + \kappa(\eta - \theta)J_\theta + \frac{1}{2}\omega^2\sigma_L^2 J_{xx} + \frac{1}{2}\sigma_I^2 y^2 J_{yy} \\ &\quad + \frac{1}{2}\nu^2\theta J_{\theta\theta} + \rho_{12}\omega\sigma_L\sigma_I y J_{xy} + \rho_{13}\omega\sigma_L\nu\sqrt{\theta}J_{x\theta} + \rho_{23}\sigma_I y\nu\sqrt{\theta}J_{y\theta}, \end{aligned} \tag{13}$$

$$\mathcal{S}_0 J = J_x - J_y, \tag{14}$$

$$\mathcal{B}_0 J = J_y - (1+\theta)J_x. \tag{15}$$

Dimension Reduction The homogeneity property of the value function (with respect to the variables x and y) allows us to use the following transformation to simplify the problem.

$$J(x, y, \theta, t) = \frac{1}{1-\gamma}(x+y)^{1-\gamma} e^{(1-\gamma)\phi(\pi,\theta,t)}, \tag{16}$$

where the scaled state variable $\pi = \frac{y}{x+y}$ is the investor's allocation to the illiquid stock and the function $\phi(\pi, \theta, t)$ can be interpreted as the certainty equivalent rate of return (CER) required

[15] For example, in the Unconstrained Case, the solution domain is $R_0 = \{(x, y, \theta, t) : \theta \geq 0,\ \ x + y - \theta y^- \geq 0,\ \ t \in [0, T]\}$, and in the Constrained Case, the solution domain is $R_1 = \{(x, y, \theta, t) : \theta \geq 0,\ \ x \geq 0,\ \ y \geq 0,\ \ t \in [0, T]\}$.

by the investor over the period $[t, T]$.[16] Let $\xi = \omega/(x + y)$ be the investor's allocation to the liquid stock. Then, it can be shown that $\phi(\pi, \theta, t)$ satisfies the following partial differential equation.

$$\max\big\{\sup_{\xi} \mathcal{L}_1^{\xi}\phi + \phi_t, \ \ -\phi_\pi, \ \ -\theta + (1 + \theta\pi)\phi_\pi\big\} = 0, \tag{17}$$

$$\phi(\pi, \theta, T) = \ln(1 - \theta\pi^-), \tag{18}$$

where the expression of operator $\mathcal{L}_1^{\xi}\phi$ is presented in the Appendix A to save space.

Optimal Trading Regions. Given the solution $\phi(\pi, \theta, t)$ to equation (17)–(18), we can define the following regions: an optimal buy region (BR),

$$\equiv \{(\pi, \theta, t) : -\theta + (1 + \theta\pi)\phi_\pi = 0\}; \tag{19}$$

an optimal sell region (SR),

$$\equiv \{(\pi, \theta, t) : -\phi_\pi = 0\}; \tag{20}$$

and an optimal no-trade region (NTR)

$$\equiv \{(\pi, \theta, t) : 0 < \phi_\pi < \frac{\theta}{1 + \theta\pi}\}. \tag{21}$$

In general, it is optimal to trade in the illiquid stock only when the wealth allocation to this stock lies in the BR or SR. The boundary between the NTR and the SR (or BR) is the optimal sell (or buy) boundary. In Section 3.2, we discuss these optimal trading boundaries in detail.

2.3 Models for Comparison Purpose

If we eliminate the randomness and predictive power of the BAS, then our model reduces to the DJL model studied in Dai et al. (2011). We consider another model that assumes

[16]Equation (16) implies that the investor is indifferent between choosing their optimal trading strategy or receiving a constant return of $\phi(\pi, \theta, t)$ over the period $[t, T]$.

Table 2: Nested Cases for Comparison

This table summarizes the nested models with the respective parameter restrictions for comparison.

Nested Model	Parameter Restrictions	Features
DJL	Set $\lambda = \kappa = \nu = 0$	Constant BAS
SBAS DJL	Set $\lambda = 0$	Random BAS but constant premium
Ours	No restriction	Random BAS and random premium

a stochastic BAS but constant expected returns on the illiquid stock to separate the effect of randomness and the predictive power of BAS. We call this model the DJL model with stochastic BAS (SBAS-DJL model).[17] Comparing our model with the SBAS-DJL model allows us to understand the effect of the predictive power of the BAS, and comparing the SBAS-DJL model with the DJL model allows us to uncover the effect of the randomness of the BAS.

We adjust the illiquid stock's (constant) expected return parameter (i.e., μ_I) in the DJL and SBAS-DJL models to its long-term mean value implied by our model (i.e., $\mu_I + \lambda\eta$) to make fair comparisons. We summarize this information in Table 2.

3 Numerical Analysis of Model Implications

Next, we conduct a quantitative analysis of the model's implications. As the models considered in this study do not permit closed-form solutions, we solve them numerically using the method outlined in the Appendix A.

Our numerical analysis has the following goals: (i) to understand the main features of the investor's optimal trading strategy; (ii) to demonstrate the importance of adopting the optimal strategy; and (iii) to examine the implications of the liquidity risk, with the provision of timing the liquidity premium, for the investor's welfare.

[17]The DJL model is obtained by setting the LPTC ratio λ, the mean reversion speed κ, and the BAS volatility ν to 0, and the SBAS-DJL model is obtained by setting λ to 0.

3.1 Model Calibration

We begin with a description of the model calibration. We create an equally weighted portfolio consisting of our sample illiquid stocks as a proxy for the illiquid stock in our model, and we calculate its effective BAS and monthly returns using the portfolio value on the bid side. We use an equally-weighted portfolio consisting of S&P 500 stocks as a proxy for the liquid stock in our model and calculate its monthly returns.[18]

Next, we use the QMLE method to obtain stock-related parameters.[19] The liquid stock has an annual expected return of $\mu_L = 12.2\%$ and a return volatility of $\sigma_L = 17.4\%$. The illiquid stock has an annual unconditional expected return of $\mu_I = 5.7\%$ and a return volatility of $\sigma_I = 18.1\%$. The correlation coefficient between the liquid stock's return and illiquid stock's return is $\rho_{12} = 0.889$. The illiquid stock's BAS has a mean value of $\eta = 1.7\%$, a mean reversion speed of $\kappa = 1.309$, and a volatility of $\nu = 7.4\%$. Notably, the loading of the expected return of the illiquid stocks on its BAS is as high as $\lambda = 7.75$ with high statistical significance, which suggests a fairly strong predictive power of the BAS. The correlation coefficient of the liquid stock's return and illiquid stock's BAS is $\rho_{13} = -0.514$, and that of the illiquid stock's return and its BAS is $\rho_{23} = -0.554$.

We estimate the risk-free rate by taking the time-series average of the Fama–French risk-free rate factor over the sample period and obtain $r = 1.2\%$. We set the coefficient of the investor's risk aversion to $\gamma = 10$.[20] The investor's investment horizon is set to $T = 5$ years.

Table 3 summarizes the default parameter values. Note that the parameter values in the DJL and SBAS-DJL models are adjusted using the formulas provided in Table 2.[21]

We set the initial value of BAS θ_0 to the long-term mean η and the investor's initial

[18] We also examine value-weighted portfolios. In this case, the unconditional expected return of the illiquid portfolio is lower and the LPTC ratio is higher, indicating that the results for the value-weighted portfolios are stronger than what we report below.

[19] To save space, we present the detailed QMLE procedure in the Appendix B.

[20] We set a relatively high risk aversion coefficient to avoid an unrealistically high level of leverage used by the investor. Assuming a smaller risk aversion coefficient will make the quantitative results even stronger.

[21] In the DJL model or SBAS-DJL model, the expected return of the illiquid stock is set to $\mu_I = 0.057 + 7.75 \times 0.017 = 0.189$, and the loading on the BAS is set to $\lambda = 0$. We also examine the results obtained by perturbing the model parameters within reasonable ranges (e.g., within $\pm$ two standard errors), and we find that the results are qualitatively similar to what we report below. These robustness test results are available from the authors upon request.

Table 3: Default Parameter Values

This table summarizes our default parameter values. For stock-related parameters, we report both their estimated values and standard errors (S.E.).

Parameter	Symbol	Estimate	S.E.
Stock-related Parameters			
Expected return of the liquid stock	μ_L	0.122	0.084
Volatility of liquid stock returns	σ_L	0.174	0.022
Unconditional expected return of the illiquid stock	μ_I	0.057	0.107
LPTC ratio of the illiquid stock	λ	7.750	1.100
Volatility of illiquid stock returns	σ_I	0.181	0.023
Correlation between liquid and illiquid returns	ρ_{12}	0.889	0.014
Correlation between liquid returns and illiquid stock BAS	ρ_{13}	-0.514	0.060
Correlation between illiquid returns and illiquid stock BAS	ρ_{23}	-0.554	0.080
Mean value of illiquid stock BAS	η	0.017	0.003
Mean reversion speed of illiquid stock BAS	κ	1.309	0.359
Volatility of illiquid stock BAS	ν	0.074	0.005
Other Parameters			
Risk-free rate	r	0.012	-
Investment horizon (years)	T	5	-
Relative risk aversion coefficient	γ	10	-

position on the illiquid stock y_{0-} to 0 unless otherwise stated.

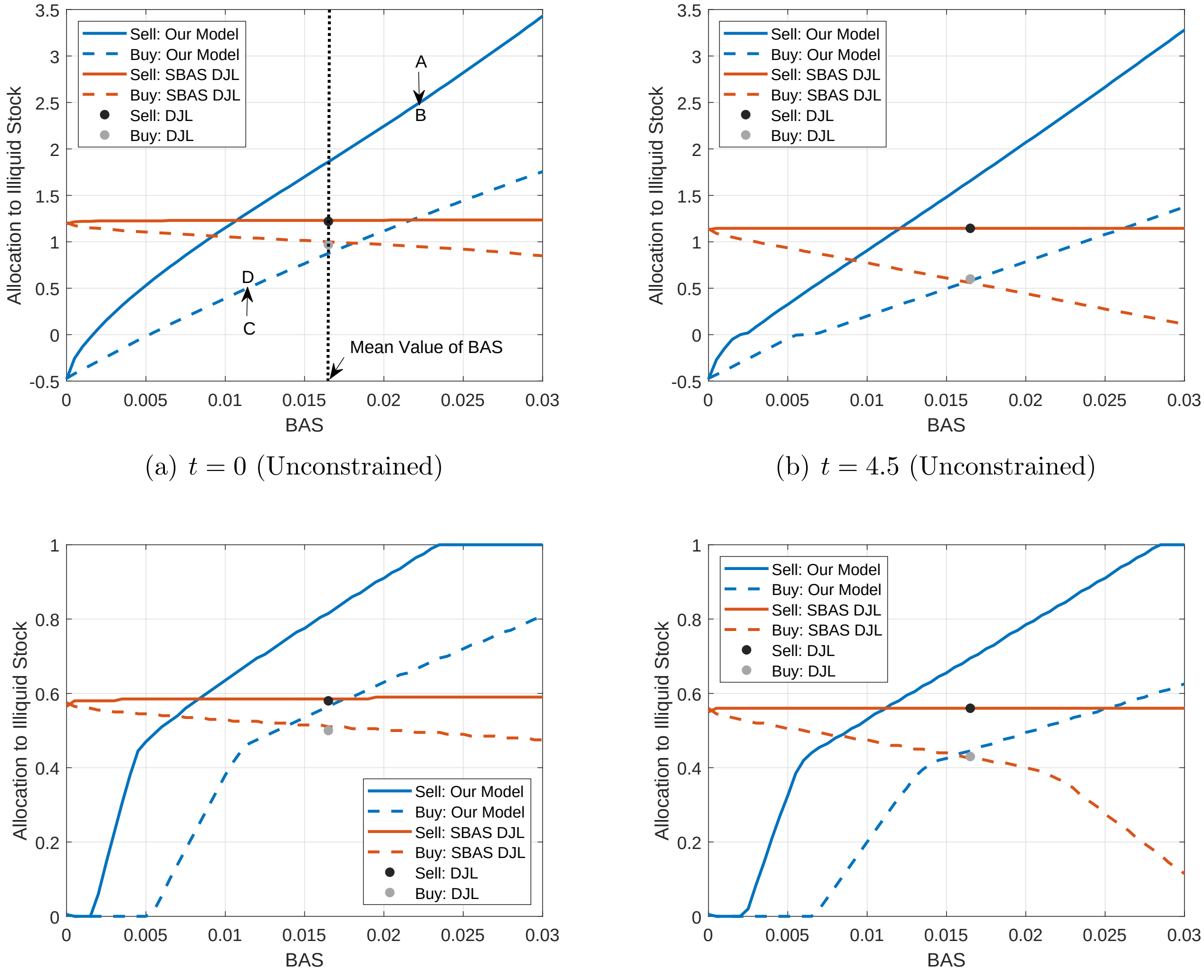
3.2 Features of the Optimal Trading Strategy

Next, we discuss the quantitative features of the investor's optimal trading strategy.

3.2.1 Optimal Trading of the Illiquid Stock

Figure 2 plots the optimal trading boundaries at time $t = 0$ and $t = 4.5$ in the DJL model, SBAS-DJL model, and our model, respectively. Subfigures (a) and (b) show the results for the Unconstrained Case, and subfigures (c) and (d) show the results for the Constrained Case.

We analyze subfigure (a) to demonstrate our main points. The optimal sell and buy boundaries are denoted by two blue lines (one solid and the other dashed) in our model, two



(a) $t = 0$ (Unconstrained)

(b) $t = 4.5$ (Unconstrained)

(c) $t = 0$ (Constrained)

(d) $t = 4.5$ (Constrained)

Figure 2: Optimal Trading Boundaries in Different Models

This figure shows the optimal trading boundaries in our model, the SBAS-DJL model, and the DJL model. Panel (a) shows the boundaries at initial time $t = 0$ and Panel (b) shows them at time $t = 4.5$ years for the case without any portfolio constraints; Panels (c) and (d) show the respective results for the cases with no-leverage and no-short-sale constraints on both stocks. The parameter values used to generate these results are reported in Table 3.

red lines in the SBAS-DJL model, and black and gray dots in the DJL model.[22] Given these boundaries, the investor's optimal trading strategy for the illiquid stock can be characterized as follows. When the investor's wealth allocation to the illiquid stock lies above the sell boundary (or below the buy boundary), it is optimal to trade in the minimum amount of illiquid stock to push its weight in the portfolio back to the sell (or buy) boundary, as signified by the arrow from point A to point B (or from point C to point D). The no-trade region lies between the sell boundary and the buy boundary. When the investor's wealth allocation to the illiquid stock lies in the no-trade region, it is better to not trade in the illiquid stock, as the benefits from rebalancing to a better risk exposure are outweighed by the trading costs incurred.

The optimal trading boundaries in our model exhibit some interesting features compared with those in the DJL and SBAS-DJL models, which we elaborate in detail below.

Capture Spot Liquidity Premium The level of both the sell and buy boundaries in our model increases with an increase in the BAS because a larger BAS implies a higher liquidity premium, which increases the investor's desired exposure to the illiquid stock. This upward shift in the optimal trading boundaries reveals the investor's incentive to earn the spot liquidity premium. However, as a larger BAS also implies greater trading costs, the no-trade region also widens as the BAS increases in the Unconstrained Case.[23] Hence, it is optimal for the investor to maintain high exposure to the illiquid stock but to reduce the frequency of rebalancing at a wide spread to earn the spot liquidity premium.[24]

We perform Monte Carlo simulations for the optimal trading strategies in the three models to better understand the dynamic properties of the optimal trading strategy. We are

[22]Recall that the BAS is constant in the DJL model. Accordingly, when generating the results in the DJL model, we fix the BAS to its mean value η.

[23]This pattern does not necessarily hold in the Constrained Case because the bindingness of the no-leverage constraint works against the choice of a wide no-trade region.

[24] The no-trade region in our model is located higher than those in the SBAS-DJL model or DJL model depending on whether the BAS is equal to its mean value η. This indicates that on average, the investor in our model desires a greater exposure to the illiquid stock. This pattern is due to the hedging demand induced by the negative correlation between shocks on the realized returns on the illiquid stock and its BAS (hence its liquidity premium). This is consistent with the results derived from other models where the expected return is predicted by alternative variables, such as dividend yield or consumption–wealth ratio (e.g., Lynch (2001) and Huang and Liu (2007)).

Table 4: Trading Statistics

This table reports the trading statistics obtained from performing 10,000 paths of Monte Carlo simulations of the optimal trading strategies implied by the three models. Panel A shows the results for the case without any portfolio constraints, and Panel B shows the results for the case with no-leverage and no-short-sale constraints on both stocks. Note that the initial trade is conducted at the average BAS η. The parameter values used in generating these results are reported in Table 3.

Panel A: Unconstrained Case

	Our Model	SBAS-DJL Model	DJL Model
Volume of Purchase below Average BAS	0.103	0.031	N.A.
Volume of Purchase above Average BAS	1.140	0.016	N.A.
Present Value of Transaction Costs	0.043	0.017	0.017

Panel B: Constrained Case

	Our Model	SBAS-DJL Model	DJL Model
Volume of Purchase below Average BAS	0.034	0.006	N.A.
Volume of Purchase above Average BAS	0.240	0.007	N.A.
Present Value of Transaction Costs	0.015	0.009	0.008

particularly interested in the volume of purchases occurring above or below the mean value of the BAS, as there is active trading for the liquidity premium when there is a large BAS. These measures help us understand the importance of timing the liquidity premium in the optimal trading strategy.

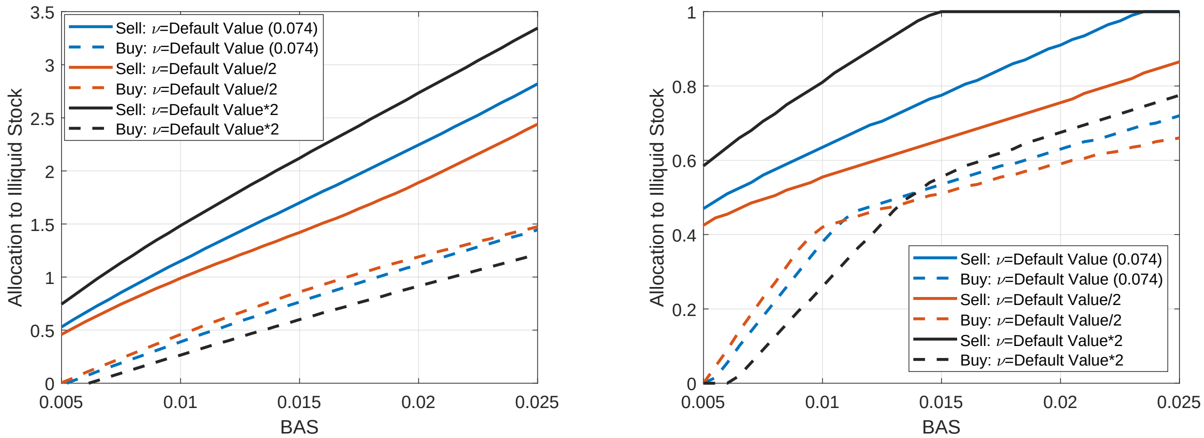
We report the trading statistics obtained from 10,000 simulated paths in Table 4. Panel A shows the results for the Unconstrained Case and Panel B shows the results for the Constrained Case. In the Unconstrained Case of our model, the volume of stock purchases above the mean BAS is much larger than those below the mean BAS (1.14 vs. 0.10), suggesting that the active timing of the liquidity premium is indeed a major component of the optimal trading strategy. In contrast, in the SBAS-DJL model, the volume of purchases below the mean BAS is much larger than those above the mean BAS because the investor only performs liquidity-timing (i.e., trades more when the spread is narrow) in the SBAS-DJL model. The results are qualitatively similar in the Constrained Case. Thus, we conclude that earning the liquidity premium is indeed an important component of the optimal trading strategy.

Control Turnover The negative correlation between the realized returns and the BAS of the illiquid stock implies that after the negative shocks on the realized returns, it is likely that the investor's exposure to the illiquid stock will decrease and the liquidity premium will increase, which increases the investor's optimal exposure to this stock. Thus, the investor has a strong incentive to purchase shares at a wide spread, but the investor should exhibit an extra incentive to reduce turnover to avoid excessive trading on wealth because it is inherently costly. A close examination of Figure 2 reveals that the optimal no-trade region in our model is much wider than those in the other two models, which confirms the above intuition.[25]

To further verify this mechanism, we note that increasing the volatility of the BAS (i.e., ν) increases the variability in the BAS and liquidity premium, which would strengthen the investor's incentive to capture the change in the liquidity premium and increase their potential trading costs. In such a case, the investor should choose a wider no-trade region to control turnover. Figure 3(a) plots the optimal no-trade region for three values of ν, which confirms that it is indeed the case.

Aiming-in-front-of-the-target Pattern Consistent with the empirical pattern, the BAS in our model is assumed to be mean-reverting. Intuitively, when the BAS is well below the mean value, it tends to increase in the near future. In this case, it is optimal for the investor to buy some stock in advance because the future expected returns are likely to increase as the BAS reverts and buying shares at the currently small BAS incurs lower trading costs. Similarly, when the BAS is well above the mean value, it tends to decrease in the near future. In this case, although the spot liquidity premium is high, reversion of the BAS reduces the future liquidity premium and the profitability of purchasing shares at the currently large BAS. Thus, the speculation on the future changes in the BAS should make the investor purchase fewer shares at a large spread. This intuition suggests that the optimal

[25] In the presence of transaction costs, active investors will choose a wider no-trade region to save on transaction costs in general. We also examine a case wherein the degree of return predictability is the same as in our model but with a constant BAS. In this case, the optimal no-trade region is narrower than that in our model. This implies that the interaction of return predictability and trading costs in our model induces the investor to further reduce turnover.



(a) Unconstrained Case (b) Constrained Case

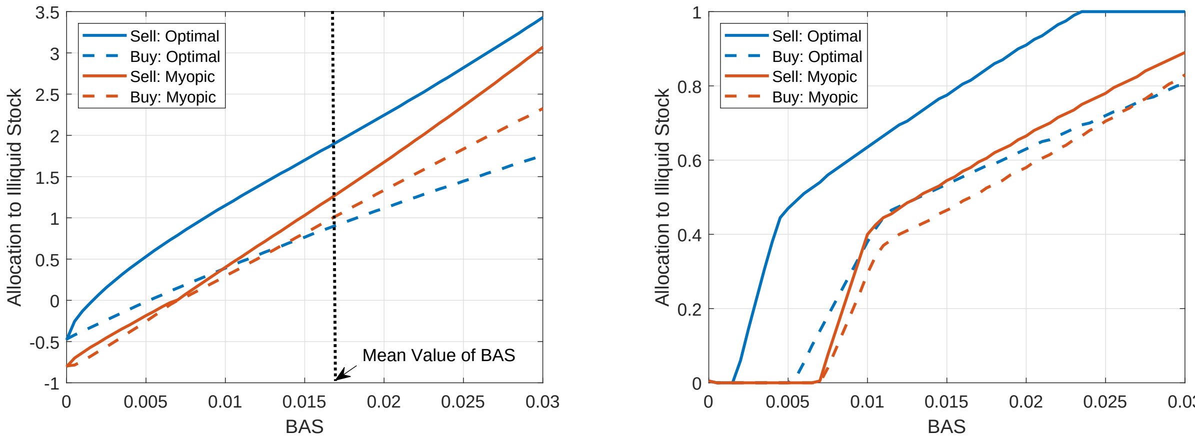
Figure 3: BAS Volatility and Optimal Trading Boundaries

This figure shows the optimal trading boundaries in our model at the initial time $t = 0$ for three values of BAS volatility (i.e., ν). Panel (a) shows the results for the case without any portfolio constraints and Panel (b) shows the results for the case with no-leverage and no-short-sale constraints on both stocks. Other parameter values used to generate these results are reported in Table 3.

trading strategy in our model should exhibit an aiming-in-front-of-the-target pattern when the current BAS is either too low or too high.

We compare the optimal trading strategy with the myopic trading strategy to verify this intuition. The myopic strategy is obtained by solving the DJL model independently for each level of the BAS. When adopting the myopic strategy, the investor takes into account the current BAS and liquidity premium but not the future changes in BAS and the liquidity premium. Figure 4 shows the optimal trading boundaries (blue lines) and the myopic trading boundaries (red lines). When the BAS is sufficiently below the mean value, the optimal buy boundary (blue dashed line) is located above the myopic buy boundary (red dashed line), implying that the investor should buy more shares at a small BAS when adopting the optimal strategy. In comparison, when the BAS is above the mean value, the optimal buy boundary is located beneath the myopic buy boundary, implying that the investor is reluctant to purchase additional shares at a large BAS when following the optimal strategy.

The aiming-in-front-of-the-target pattern in our model is driven by the mean reversion property of the BAS. If we increase the mean reversion speed, then the investor should have more of an incentive to buy shares when the BAS is narrow and less of an incentive



(a) Unconstrained Case

(b) Constrained Case

Figure 4: Optimal vs. Myopic Trading Boundaries

This figure shows the optimal and myopic trading boundaries at initial time $t = 0$ in our model. When generating the myopic trading boundaries, the investor ignores the future variations in the BAS by incorrectly assuming $\kappa = \nu = 0$. Panel (a) shows the results for the case without any portfolio constraints and Panel (b) shows the results for the case with no-leverage and no-short-sale constraints on both stocks. Other parameter values used in generating these results are reported in Table 3.

to buy when the BAS is large, as a low/high level of the BAS and liquidity premium are less persistent. To verify this intuition, Figure 5(a) plots the optimal trading boundaries for three values of mean reversion speed. We find that when the BAS is well below (or above) the mean value, the optimal buy boundary moves north (or south) as the mean reversion speed increases, indicating a higher (or lower) propensity to buy stock.

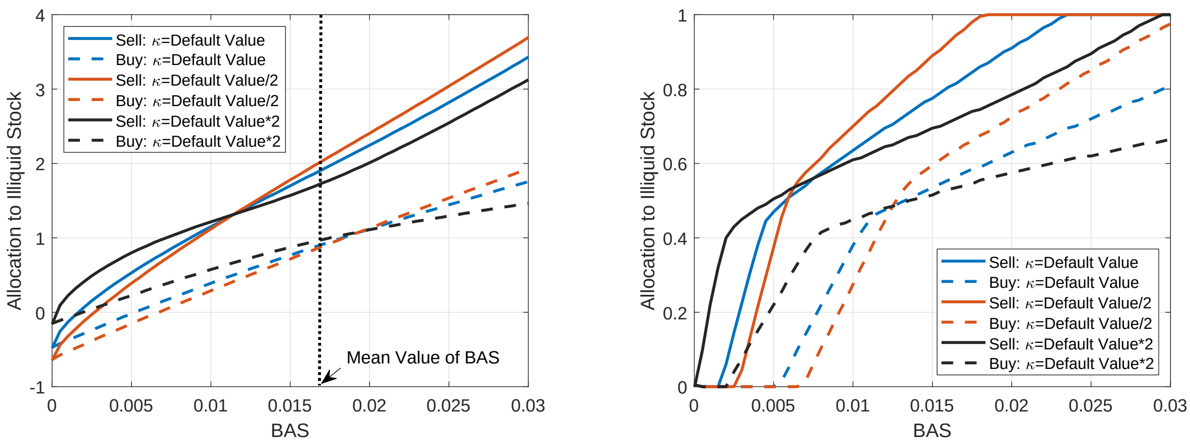
3.3 Losses from Adopting Suboptimal Trading Strategies

Next, we demonstrate the economic importance of adopting the optimal trading strategy by calculating the investor's utility losses when following suboptimal trading strategies.

For a given strategy Π, the investor's utility loss from adopting Π is defined as the solution Δ_Π to equation

$$V(1 - \Delta_\Pi, 0, \theta_0, 0) = V^\Pi(1, 0, \theta_0, 0), \tag{22}$$

where $V^\Pi(x, y, \theta, t)$ is the indirect utility function associated with strategy Π. In other words, Δ_Π is the certainty equivalent wealth loss (CEWL) as a fraction of the investor's initial level



(a) Unconstrained Case

(b) Constrained Case

Figure 5: Mean Reversion Speed and Optimal Trading Boundaries

This figure shows the optimal trading boundaries in our model, at initial time $t = 0$, for three values of the mean reversion speed of the BAS (i.e., κ). Panel (a) shows the results for the case without any portfolio constraints and Panel (b) shows the results for the case with no-leverage and no-short-sale constraints on both stocks. Other parameter values used in generating these boundaries are reported in Table 3.

of wealth from adopting strategy Π.

3.3.1 Loss from Adopting the Myopic Strategy

We first examine the investor's loss from adopting the myopic trading strategy instead of the optimal one. When adopting the myopic trading strategy, although the investor manages to dynamically time the liquidity premium, they do not optimally control the turnover.[26] Consequently, the investor is likely to incur high transaction costs and suffer a utility loss.

Figure 6 plots the investor's average trading costs and CEWL Δ_1 against BAS volatility (i.e., ν). We find that adopting the myopic strategy leads to a significant utility loss for the investor. In the base case, Δ_1 is as high as 12.2% (or 3.3%) of the investor's initial wealth in the Unconstrained Case (or Constrained Case). Moreover, Δ_1 increases with an increase in BAS volatility because a higher BAS volatility implies greater variability in the liquidity premium, which implies the importance of optimally trading off between the

[26]The myopic trading boundaries are shown in Figure 4, which suggests that the no-trade region implied by the myopic trading strategy is much narrower than that implied by the optimal strategy.

liquidity premium and transaction costs.

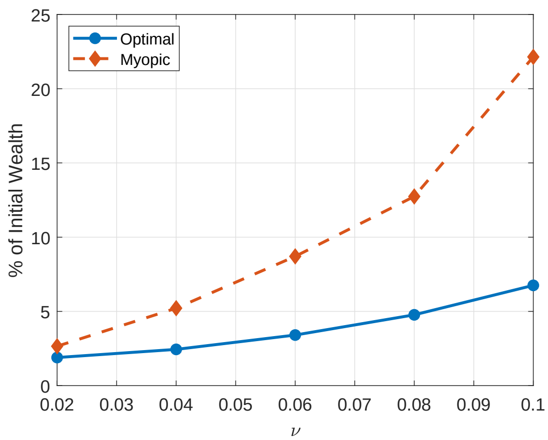
A distinct feature of our portfolio choice problem is that there is a strong interaction between return predictability and trading costs. We argue that this interaction drives the large utility loss from following the myopic strategy. For this purpose, we calculate the investor's CEWL from adopting the myopic strategy in a model with the same degree of return predictability but without transaction costs, and we denote this CEWL as δ_1.[27] Figure 7 shows the CEWL ratio Δ_1/δ_1. The CEWL in our model is usually more than 2–3 times greater than that in the model without trading costs. This suggests that adopting the optimal trading strategy is especially important in the liquidity premium-timing problem.[28]

Comparative Statics In Table 5, we report the comparative statics results for the utility loss from adopting the myopic trading strategy.

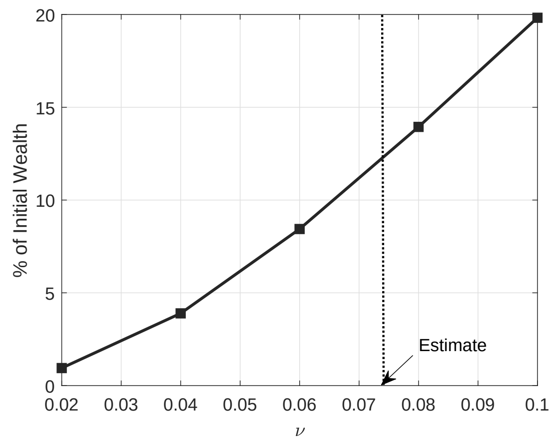
First, the reported results suggest that for a large range of parameter values, the investor incurs a substantial loss from adopting the myopic trading strategy. Moreover, we observe some interesting patterns in Table 5. For example, the CEWL decreases with an increase in the illiquid stock's unconditional expected return μ_I but increases with an increase in the liquid stock's expected return μ_L. As the illiquid stock's unconditional expected return decreases relative to the liquid stock's expected return, increased trading in the illiquid stock is driven by the changes in its liquidity premium. Consequently, the CEWL from not timing the liquidity premium optimally increases. As the investor's risk aversion coefficient or risk-free rate decreases, the CEWL increases because the investor allocates more wealth to the illiquid stock. The CEWL also increases as the investor's investment horizon increases because a longer horizon implies more opportunities to actively time the liquidity premium, which increases the importance of optimal trading. Last, the CEWL decreases with a decrease in the mean reversion speed of the BAS because a faster mean reversion implies shorter investment periods with a high liquidity premium, and consequently, the value of

[27]In such a model, the stock price dynamics are the same as those in our baseline model but the investor does not pay costs on trading.

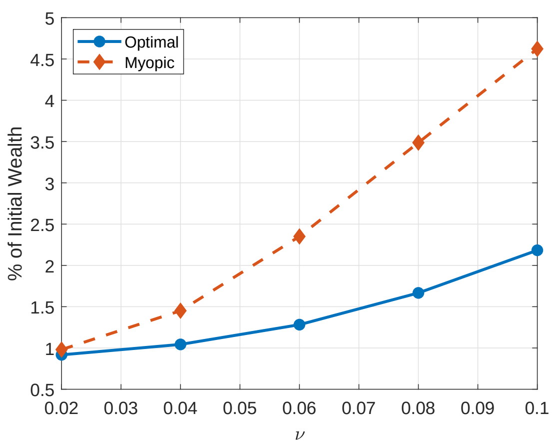
[28]In addition, we examine the utility loss from adopting the myopic trading strategy in two alternative models: the SBAS-DJL model where there are stochastic trading costs but no return predictability and a model with time-varying expected returns but a constant BAS. We find that the CEWLs in these two models are also smaller than those in our model.



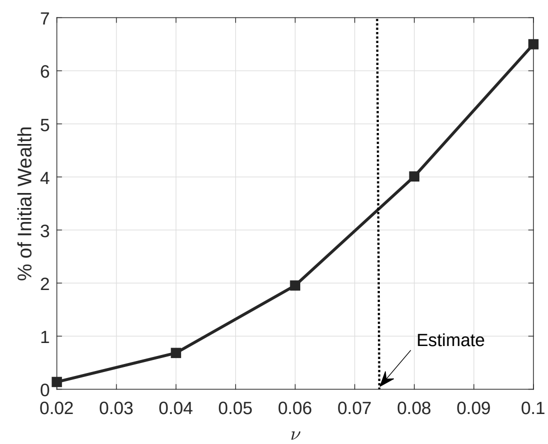
(a) PVTC: Unconstrained Case



(b) CEWL: Unconstrained Case



(c) PVTC: Constrained Case



(d) CEWL: Constrained Case

Figure 6: Trading Costs and CEWL from Adopting the Myopic Strategy

This figure shows the transaction costs incurred by the investor when adopting the optimal trading strategy or the myopic trading strategy (subfigures on the left) and the CEWL from adopting the myopic trading strategy (subfigures on the right). Panels (a) and (b) show the results for the case without any portfolio constraints, and Panels (c) and (d) show the results for the case with no-leverage and no-short-sale constraints on both stocks. Other parameter values used in generating these results are reported in Table 3.

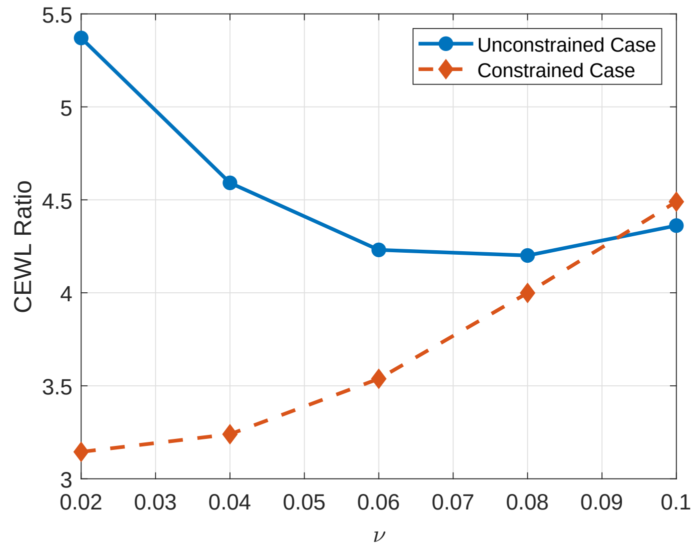


Figure 7: Ratio of CEWLs with/without Trading Costs

This figure shows the ratio of the investor's CEWL from adopting the myopic trading strategy with and without transaction costs. When generating the results without trading costs, we assume that the BAS of the illiquid stock (i.e., θ_t) continues to predict the future stock returns, but the investor does not need to exceed the spread to purchase shares (i.e., the transaction costs are waived). Other parameter values used in generating these results are reported in Table 3.

Table 5: CEWL from Adopting the Myopic Strategy: Comparative Statics

This table reports the investor's average trading costs and CEWL from adopting the myopic trading strategy for various sets of parameter values. The parameter values in the base case are reported in Table 3.

	Unconstrained Case			Constrained Case		
	TC (Optimal)	TC (Myopic)	CEWL	TC (Optimal)	TC (Myopic)	CEWL
Base case	0.043	0.116	0.122	0.015	0.031	0.033
$\gamma = 6$	0.089	0.222	0.193	0.019	0.048	0.051
$T = 10$	0.107	0.338	0.251	0.027	0.075	0.076
$r = 0.005$	0.044	0.117	0.125	0.016	0.033	0.035
$r = 0.02$	0.042	0.114	0.120	0.015	0.030	0.031
$\mu_L - 1\%$	0.048	0.121	0.115	0.015	0.029	0.029
$\mu_L + 1\%$	0.039	0.115	0.131	0.016	0.036	0.040
$\mu_I - 1\%$	0.039	0.111	0.127	0.015	0.034	0.037
$\mu_I + 1\%$	0.048	0.123	0.118	0.016	0.030	0.031
$\lambda \times 0.8$	0.025	0.080	0.095	0.012	0.029	0.029
$\lambda \times 0.6$	0.011	0.050	0.069	0.009	0.026	0.032
$\kappa \times 0.8$	0.051	0.118	0.124	0.017	0.032	0.037
$\kappa \times 1.2$	0.039	0.115	0.122	0.014	0.031	0.031
$\eta \times 0.8$	0.027	0.080	0.099	0.012	0.028	0.032
$\eta \times 1.2$	0.072	0.169	0.147	0.020	0.037	0.040

optimal timing decreases.

3.3.2 Loss from Ignoring the Predictive Power of the BAS

Studies on portfolio choice with the BAS do not consider the predictive power of the BAS on future stock returns. In our model, if the investor erroneously assumes that the illiquid stock's expected return is constant (e.g., its long-term mean value $\mu_I + \lambda\eta$), then their portfolio choice strategy will not respond to actual changes in the illiquid stock's expected return, which will cause a utility loss.

Figure 8 shows the investor's annual CEWL from erroneously assuming a constant expected return of $\mu_I + \lambda\eta$ for the illiquid stock. The resultant utility loss is also significant, especially when the investor has a long investment horizon and loose portfolio constraints. For example, with an investment horizon of 10 years, an unconstrained investor would incur an annual CEWL of 0.96% if they ignore the return predictability derived from the BAS of the illiquid stock; if the investor were subject to strict no-leverage and no-short-sale constraints, then the annual CEWL would decrease to 0.45% because of less investment in the illiquid stock. These utility loss calculations clearly demonstrate that ignoring the return predictability derived from the BAS of the illiquid stock is costly.

3.3.3 Simulation Using Market Returns

We examine the performance of three strategies to further demonstrate the value of adopting the optimal strategy: the optimal strategy, the myopic strategy, and the strategy without the liquidity premium timing, using the sample data constructed in Section 2.1 and Section 3.1. The investment horizon consistent with our sample period is 20 years. Because the BAS data are constructed with a monthly frequency, we let the investor make the trading decisions on a monthly basis.[29]

Figure 9 shows the investor's wealth accumulation processes with an initial unit wealth level. Panel (a) ((b)) shows the results for the Unconstrained Case (Constrained Case). We

[29] As in our numerical analysis, we assume a risk aversion coefficient of 10 when generating the trading strategies. We also consider other values of a risk aversion coefficient, and the results are qualitatively similar to what we report here.

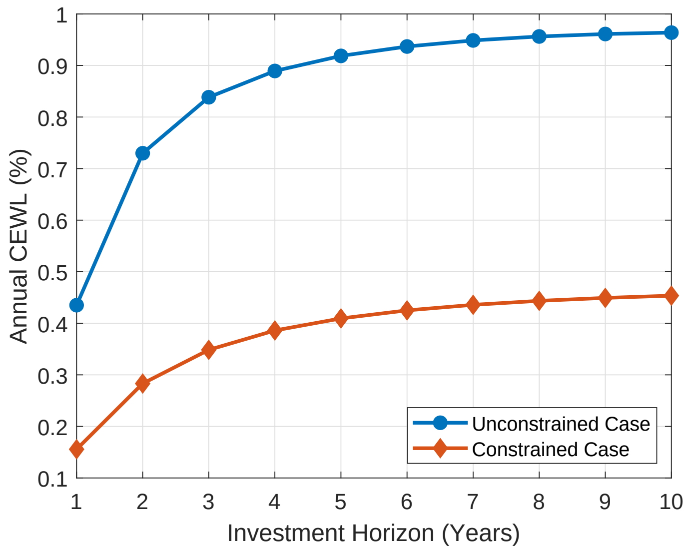
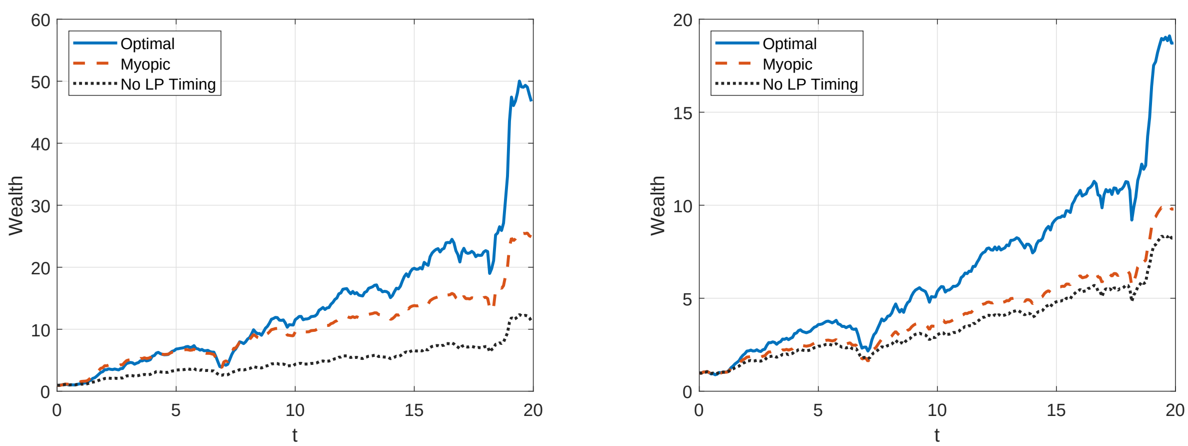


Figure 8: Annual CEWL from Ignoring the Predictive Power of the BAS

This figure shows the investor's annual CEWL from ignoring the predictive power of the BAS of the illiquid stock. The parameter values used in generating these results are reported in Table 3.



(a) Unconstrained Case

(b) Constrained Case

Figure 9: Wealth Process

This figure shows the investor's wealth accumulation process generated by adopting the optimal strategy, myopic strategy, or strategy without the liquidity premium timing. Panel (a) shows the results for the case without any portfolio constraints, and Panel (b) shows the results for the case with no-leverage and no-short-sale constraints on both stocks. The investor's initial wealth level is normalized to 1. The parameter values used in generating trading strategies are reported in Table 3.

Table 6: Empirical Performance Measures

This table reports the performance measures obtained by implementing the optimal trading strategy, myopic trading strategy, or the no-liquidity premium-timing strategy. Panel A reports the results for the case without any portfolio constraints, and Panel B reports the results for the case with no-leverage and no-short-sale constraints on both stocks. The parameter values used in generating trading strategies are reported in Table 3.

Panel A: Unconstrained Case

	Optimal Strategy	Myopic Strategy	No LP Timing
Present Value of Transaction Costs	0.976	2.336	0.030
Average Monthly Return (%)	1.70	1.33	1.07
Volatility of Monthly Returns (%)	5.22	4.75	3.27
Sharpe Ratio (%)	30.71	25.82	29.71

Panel B: Constrained Case

	Optimal Strategy	Myopic Strategy	No LP Timing
Present Value of Transaction Costs	0.163	0.214	0.019
Average Monthly Return (%)	1.30	0.99	0.93
Volatility of Monthly Returns (%)	4.13	3.49	2.87
Sharpe Ratio (%)	29.13	25.62	28.80

find that adopting the optimal trading strategy generates higher wealth levels than the other two trading strategies in both cases, which indicates the profitability of timing liquidity in practice.

Table 6 reports the performance measures obtained from this exercise, including the present value of transaction costs, average monthly portfolio returns, standard deviation of monthly returns, and Sharpe ratio. These results show that the investor incurs excessive transaction costs from adopting the myopic strategy, which lower the investment returns and Sharpe ratio, and that ignoring the liquidity premium timing makes the investor trade too conservatively, which also lowers the investment returns and Sharpe ratio.

We also examine the economic value of timing the liquidity premium under different market conditions. We consider the following five subperiods: January 2002–December 2002 (when the dot-com bubble burst), January 2003–December 2007, January 2008–December 2009 (the global financial crisis), January 2010–December 2019, and January 2020–December 2020 (the COVID-19 pandemic) to do so. The average BAS in the five subperiods are 3.03%,

Table 7: Performance Measures: Subperiod Analysis

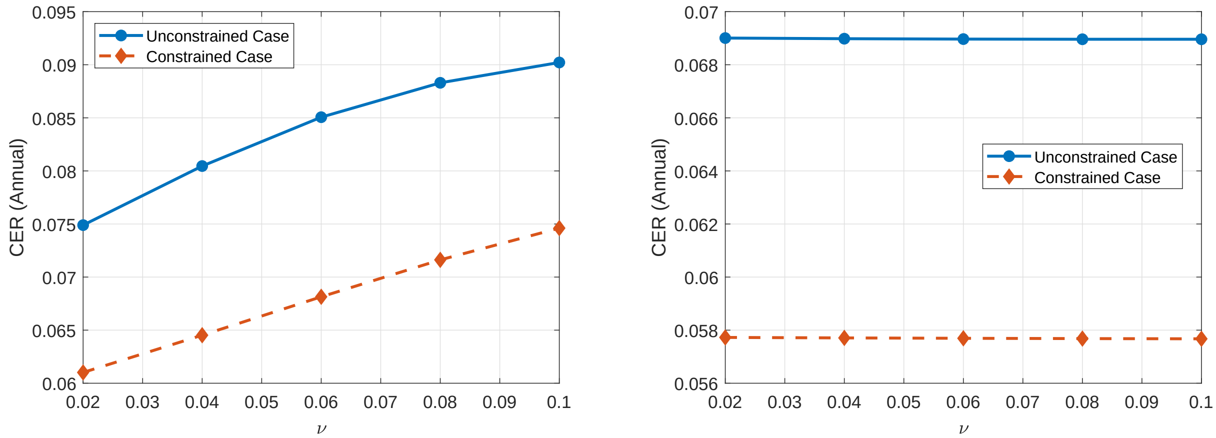
This table reports the performance measures in different subperiods, obtained by implementing the optimal timing strategy or the no-timing strategy. Panel A reports the results for the case without any portfolio constraints, and Panel B reports the results for the case with no-leverage and no-short-sale constraints on both stocks. Row i reports the results obtained for the ith subperiod for $i = 1, ...5$. The parameter values used in generating trading strategies are reported in Table 3.

		Panel A: Unconstrained Case		**Panel B: Constrained Case**	
		Optimal Timing	No Timing	Optimal Timing	No Timing
1	Avrg. Monthly Rtn (%)	2.18	1.12	0.40	0.24
	Sharpe Ratio (%)	32.41	34.17	5.45	4.59
2	Avrg. Monthly Rtn (%)	2.87	1.87	2.10	1.43
	Sharpe Ratio (%)	56.18	54.63	54.58	53.10
3	Avrg. Monthly Rtn (%)	1.13	0.35	0.83	0.26
	Sharpe Ratio (%)	11.81	6.14	9.93	3.79
4	Avrg. Monthly Rtn (%)	0.87	0.61	0.89	0.72
	Sharpe Ratio (%)	26.36	22.68	29.02	28.72
5	Avrg. Monthly Rtn (%)	3.95	2.64	2.64	1.81
	Sharpe Ratio (%)	41.15	41.59	33.59	28.44

1.63%, 2.67%, 1.43%, and 1.94%, respectively.[30]

We compare the performance of the optimal timing strategy and the no-timing strategy for these subperiods and report the results in Table 7. We find that the optimal timing strategy largely outperforms the no-timing strategy during periods of market turmoil, such as the 2008-2009 global financial crisis. Notably, in the Constrained Case, the performance advantage of the optimal timing strategy is almost entirely concentrated in periods with high BAS. In the Unconstrained Case, the optimal timing strategy delivers higher returns but lower Sharpe ratios in the first and fifth subperiods. This could be due to a high leverage ratio implied by the strategy in the absence of portfolio constraints.

[30] The differences in these average BAS is statistically significant. For example, the difference between the average BAS in the second period and that in the first period has a t-statistic of -8.58.



(a) Our Model (b) SBAS DJL Model

Figure 10: BAS Volatility and Investor's CER

This figure shows the investor's CER for various values of the BAS volatility (i.e., ν). Panel (a) shows the results for our model, and Panel (b) shows the results for the SBAS-DJL model. Other parameter values used in generating these results are reported in Table 3.

3.4 Is Higher Liquidity Risk a Curse?

In our model, the investor has an option to dynamically time both the liquidity and the liquidity premium. Holding everything else constant, if the BAS of the illiquid stock becomes more volatile, the value of such an option should be higher as the investor has more opportunities to exercise their timing. This intuition suggests that a higher liquidity risk, in the form of greater uncertainty in the future BAS, may make the investor better off.

To verify the above intuition, we examine how the investor's utility changes with respect to the volatility of the BAS (i.e., ν). In Panel (a) of Figure 10, we plot the investor's CER against ν. We find that the investor's CER increases with an increase in ν, which indicates that greater uncertainty in the BAS can indeed benefit the investor.[31]

Note that our model allows the investor to perform both liquidity timing (i.e., trade more when the spread is narrow) and liquidity premium timing (i.e., rebalance positions in response to changes in the liquidity premium). To distinguish between the effect of each channel, in Panel (b) of Figure 10, we plot the investor's CER against ν in the SBAS-DJL

[31] We emphasize that timing the optimal liquidity premium is important for the investor to benefit from greater variability in the BAS. If the investor trades sub-optimally (e.g., by adopting the myopic strategy), then their CER will not increase with an increase in ν.

Table 8: BAS Volatility and CER: Comparative Statics

This table reports the investor's annual CER for various sets of parameter values. The parameter values in the base case are reported in Table 3.

	Unconstrained Case			Constrained Case		
BAS volatility ν	0.02	0.06	0.10	0.02	0.06	0.10
Base case	0.075	0.085	0.090	0.061	0.068	0.075
$\gamma = 6$	0.116	0.133	0.141	0.094	0.103	0.109
$T = 10$	0.077	0.088	0.092	0.062	0.070	0.077
$r = 0.005$	0.071	0.082	0.088	0.058	0.066	0.073
$r = 0.02$	0.079	0.088	0.093	0.065	0.071	0.077
$\mu_L - 1\%$	0.082	0.092	0.096	0.061	0.068	0.075
$\mu_L + 1\%$	0.069	0.079	0.086	0.061	0.068	0.075
$\mu_I - 1\%$	0.064	0.074	0.080	0.055	0.062	0.068
$\mu_I + 1\%$	0.087	0.098	0.102	0.067	0.075	0.082
$\lambda \times 0.8$	0.050	0.056	0.062	0.047	0.051	0.055
$\lambda \times 0.6$	0.036	0.038	0.041	0.036	0.038	0.040
$\kappa \times 0.8$	0.076	0.087	0.091	0.062	0.070	0.077
$\kappa \times 1.2$	0.074	0.083	0.089	0.061	0.067	0.073
$\eta \times 0.8$	0.051	0.058	0.065	0.047	0.052	0.058
$\eta \times 1.2$	0.108	0.120	0.122	0.077	0.086	0.094

model, in which the investor can only perform liquidity timing because the expected return of the illiquid stock is constant. The results suggest that the CER tends to decrease slightly in ν in the SBAS-DJL model. Hence, we conclude that in our model, the utility gain from greater uncertainty in the BAS is solely attributed to more opportunities to time the liquidity premium.

In Table 8, we report the investor's annual CER for various sets of parameter values. The result that a greater BAS volatility makes the investor better off holds for a large range of parameter values.

3.5 Optimal Liquidity Premium Timing under Relative Performance Concern

In practice, many equity fund managers use portfolios of liquid assets (e.g., S&P 500) as benchmarks for performance evaluation. The reputation, status, and compensation of fund managers are closely linked to the performance of their managed portfolios against the benchmark (see, e.g., Ma et al. (2019)). For such fund managers, adopting the liquidity premium timing strategy may drive their portfolios away from the benchmarks and can be inherently risky. Therefore, the presence of relative performance concerns can affect managers' incentives to time the liquidity premium. Therefore, we next examine the optimal liquidity premium timing under the concerns of relative performance.

3.5.1 Modeling Relative Performance Concerns

We extend our baseline model in a straightforward way to incorporate relative performance concerns. The investment opportunity set is the same as that in the baseline model, and we assume that a fund manager uses the liquid stock as the performance benchmark. We capture the manager's relative performance concerns by assuming the manager has the following objective function.

$$\max E\left[u\left(W_T\left(\frac{W_T/W_0}{S_{LT}/S_{L0}}\right)^p\right)\right], \tag{23}$$

where $p \geq 0$ denotes the importance of relative performance: the higher the value of p, the more important the relative performance is to the fund manager.[32] As only the returns matter for the relative performance term (i.e., $\frac{W_T/W_0}{S_{LT}/S_{L0}}$) in (23), without loss of generality, we can set $W_0 = S_{L0}$ such that the objective function becomes as follows

$$\max E\left[u\left(W_T\left(\frac{W_T}{S_{LT}}\right)^p\right)\right]. \tag{24}$$

To save space, we provide the details of this model in the Appendix C. This model is solved numerically with the default parameter values reported in Table 3.

[32]The interpretation of (23) is straightforward—when the manager's portfolio outperforms (or underperforms) the benchmark, the manager can obtain extra utility (or disutility) in addition to that derived from the asset under their management. Note that the specification of relative performance concerns in (23) does not create convexity in the manager's optimization problem, which tends to penalize deviations from the benchmark. In contrast, if relative performance concerns are specified in a way that creates convexity (e.g., Basak et al. (2007)), then deviations from the liquid benchmark will be preferred and our results are expected to be stronger.

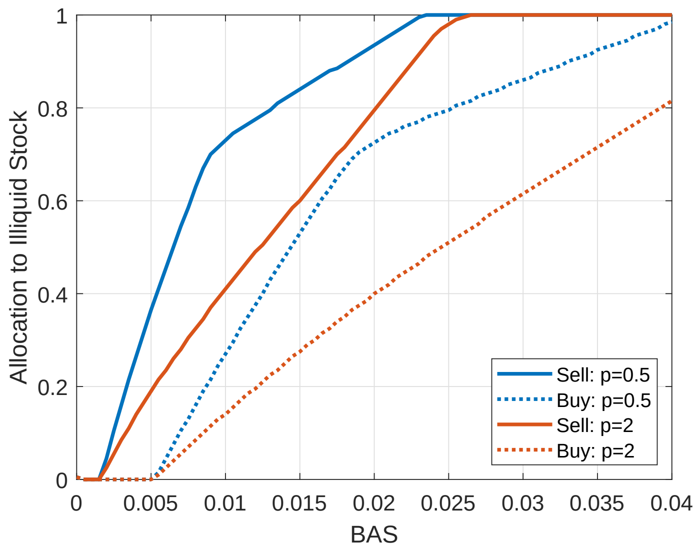


Figure 11: Optimal Trading Boundaries under Relative Performance Concerns
This figure shows the fund manager's optimal trading boundaries for two values of the relative performance concern parameter (i.e., p). Other parameter values used in generating these results are reported in Table 3.

3.5.2 Analysis of Results

Next, we conduct a numerical analysis of the solution to the fund manager's problem. In practice, there are borrowing and short-selling constraints in the mutual fund industry (see, e.g., Almazan et al. (2004)). Accordingly, we assume that the fund manager is restricted from borrowing or short-selling when generating these results.

Optimal Trading Strategy Figure 11 shows the fund manager's optimal trading boundaries for two values of p. The relative performance concerns indeed have an impact on the optimal trading boundaries: as the value of p increases, the level of both the optimal buy and sell boundaries decreases, which implies that the optimal exposure to the illiquid stock decreases with a decrease in the degree of relative performance concerns. This pattern reveals the manager's stronger aversion to deviations from the benchmark stock in their managed portfolio.

Costs of Following Suboptimal Strategies Next, we calculate the fund manager's utility loss from adopting the myopic trading strategy or the strategy without liquidity premium timing to understand the importance of adopting the optimal trading strategy in the presence of relative performance concerns. Similar to the analysis in Section 3.3, we measure utility costs as a fraction of the fund's initial assets under management (AUM).

Figure 12 shows the utility losses against the value of p. We find several interesting observations from it. First, in the presence of relative performance concerns, the costs of suboptimal trading can continue to be economically significant. For example, when $p = 1.5$, the cost of erroneously assuming constant expected returns on the illiquid stock amounts to approximately 1.9% of the fund's initial AUM, and the cost increases to 6.1% if the fund manager follows the myopic trading strategy. Second, the losses appear to be hump-shaped functions of p. The intuition behind this pattern is as follows. A higher degree of relative performance concerns have two competing effects: it can cause greater marginal utility, which increases the costliness of suboptimal trading, and it drives the manager's portfolio (both optimal and suboptimal) toward the benchmark liquid stock, which decreases the costliness of suboptimal trading. These results suggest that when the value of p is small, the first effect dominates and makes the utility costs increase with an increase in p, and when the value of p is large, the second effect dominates and makes the utility costs decrease with a decrease in p instead.

Overall, the above results indicate that a moderate degree of relative performance concerns increase the importance of timing the optimal liquidity premium.

BAS Volatility and Investor's Utility In Section 3.4, we have shown that with optimal liquidity premium timing, a greater variability in the BAS can make the investor better off. Here, we examine whether this result holds in the presence of relative performance concerns.

We first consider the fund manager's utility. Figure 13 plots the fund manager's scaled value function against the BAS volatility (i.e., ν) for three values of p.[33] We find that the scaled value function is increasing with an increase in the volatility of the BAS when relative performance concerns are present.

Next, we calculate the utility derived by a fund investor who exhibits no relative perfor-

[33]This scaled value function is defined as the function ϕ in (51) in the Appendix C.

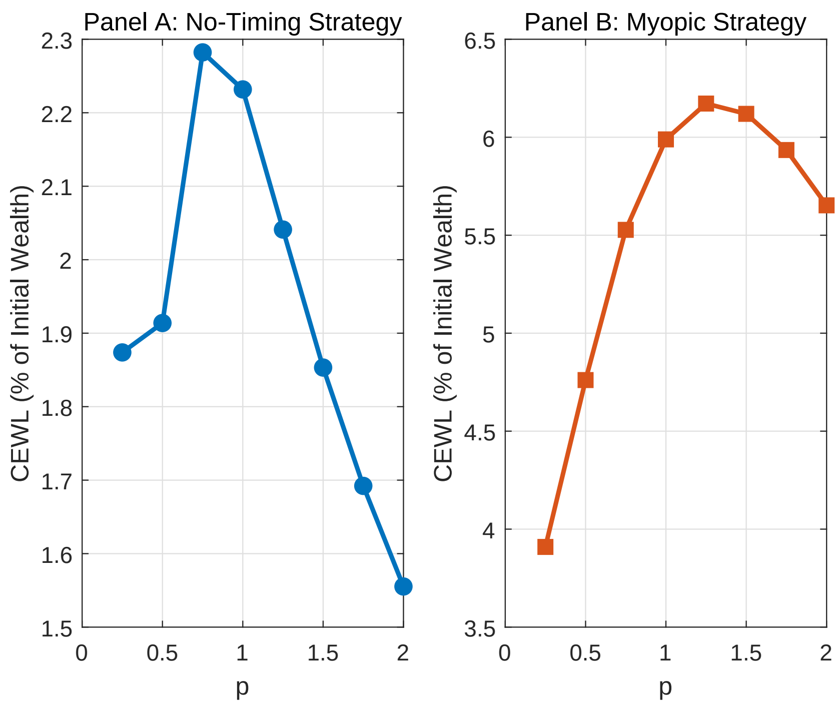


Figure 12: Loss from Suboptimal Trading under Relative Performance Concerns

This figure shows the fund manager's utility loss from adopting the strategy without liquidity premium timing (the no-timing strategy) or the myopic strategy against the value of the relative performance concern parameter (i.e., p). The no-timing strategy is obtained by assuming a constant expected return on the illiquid stock and the myopic strategy is obtained by assuming no future variation in the BAS. Other parameter values used in generating these results are reported in Table 3.

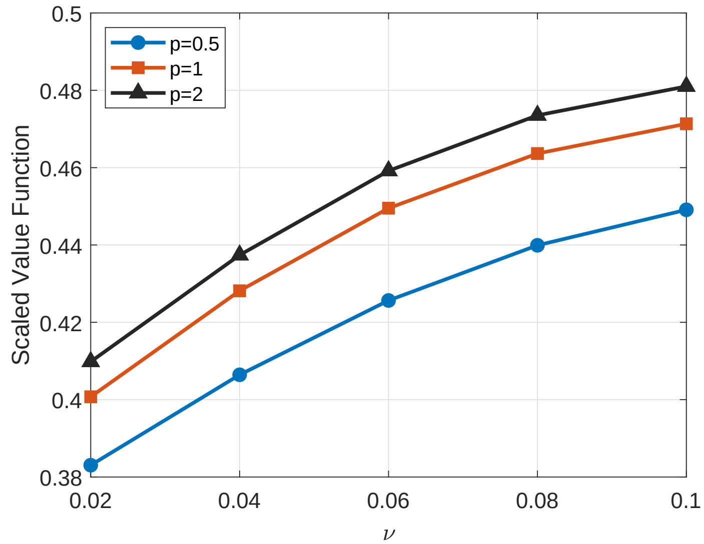


Figure 13: BAS Volatility and Fund Manager's Scaled Value Function

This figure shows the fund manager's scaled value function against the volatility of the BAS (i.e., ν) for three values of the relative performance concern parameter (i.e., p). Other parameter values used in generating these results are reported in Table 3.

mance concern and delegates investment management to the fund manager.[34] We assume that the fund investor has a relative risk aversion coefficient of γ_I, which may differ from the manager's risk aversion coefficient γ. Note that we calculate the fund investor's expected utility using the fund manager's optimal trading strategy as input.

Figure 14 plots the fund investor's annualized CER against the volatility of the BAS for various values of p and γ_I. In all cases reported in this figure, the fund investor's CER is increasing with an increase in the value of ν. Hence, a greater variability in the BAS makes the fund investor better off as well.

4 Conclusion

We study an optimal portfolio choice problem in which the BAS of illiquid stock evolves randomly and predicts its future returns. Compared with the traditional portfolio choice models with transaction costs, there is a strong interaction between return predictability and trading costs in our model, as both are derived from the BAS. We show that this

[34]We explain the calculation of the fund investor's utility in the Appendix C.

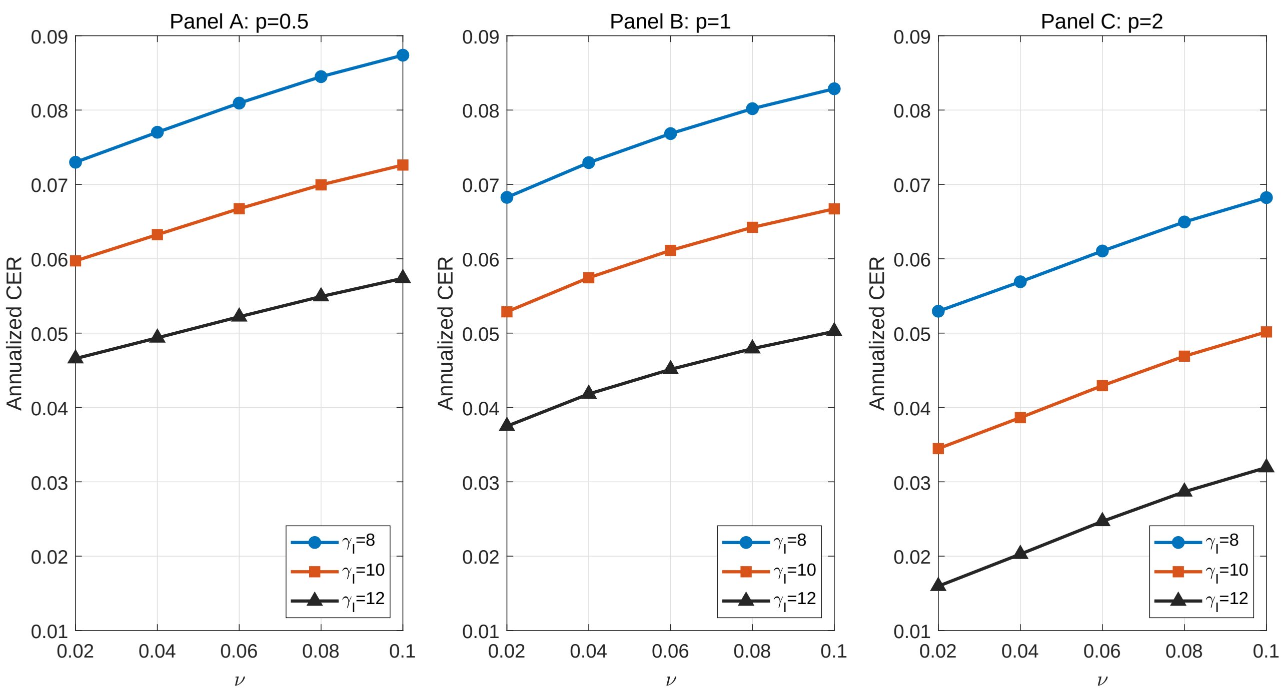


Figure 14: BAS Volatility and Fund Investor's CER

This figure shows the fund investor's annual CER against the volatility of the BAS (i.e., ν) for different values of the relative performance concern parameter (i.e., p) and the fund investor's risk aversion coefficient (i.e., γ_I). Other parameter values used in generating these results are reported in Table 3.

interaction leads to prominent features in the optimal trading strategy. First, the optimal trading strategy involves a combination of liquidity premium timing and turnover control. The investor in our model makes an effort to reduce turnover and trading costs by choosing a much wider no-trade region. Second, when the spot spread deviates sufficiently from its mean value, it is optimal to adopt an aiming-in-front-of-the-target trading strategy that allows the investor to speculate on future changes in the spread.

We also argue that such interaction makes it particularly important to adopt the optimal trading strategy because the investor incurs a significant utility loss from adopting a myopic trading strategy. Ignoring the predictive power of spread is also quite costly. We further demonstrate that a higher liquidity risk, in the form of greater uncertainty in future changes in the spread, can make the investor better off if they optimally time the liquidity premium.

We further study an optimal liquidity premium timing problem of a fund manager who is averse to poor performance against a liquid benchmark. We show that although such relative performance concerns reduce the manager's incentive to earn the liquidity premium, they can still generate economically significant gains by timing the optimal liquidity premium. Moreover, when the fund manager times the optimal liquidity premium, the greater variability in the BAS makes both the fund manager and fund investor better off.

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Appendix

The content of this Appendix is as follows. In Appendix A, we present the complete form of the HJB equation in the baseline model after dimension reduction and explain how we solve it numerically. In Appendix B, we present the detailed procedure of the quasi-maximum likelihood estimation (QMLE) method used for estimating the model parameter values. In Appendix C, we present the details of the model studied in Section 3.5.

A Details of Solving the Baseline Model

First, it is not difficult to derive that the differential operator $\mathcal{L}_1^\xi$ in (17) is

$$\begin{aligned}\mathcal{L}_1^\xi \phi &= D + D_\pi \phi_\pi + D_\theta \phi_\theta + D_{\pi\pi}(\phi_{\pi\pi} + (1-\gamma)\phi_\pi^2) + D_{\theta\theta}(\phi_{\theta\theta} + (1-\gamma)\phi_\theta^2) \\ &\quad + D_{\pi\theta}(\phi_{\pi\theta} + (1-\gamma)\phi_\pi\phi_\theta), \end{aligned} \tag{25}$$

where the coefficients are as follows:

$$D = r(1-\pi) + \xi(\mu_L - r) + (\mu_I + \lambda\theta)\pi - \frac{1}{2}\gamma\xi^2\sigma_L^2 - \frac{1}{2}\gamma\sigma_I^2\pi^2 - \gamma\rho_{12}\xi\sigma_L\sigma_I\pi; \tag{26}$$

$$\begin{aligned}D_\pi &= (\mu_I + \lambda\theta - r)\pi(1-\pi) - \pi\xi(\mu_L - r) - \pi(1-\pi)\gamma\sigma_I^2\pi + \pi\gamma\xi^2\sigma_L^2 \\ &\quad - \pi\gamma\rho_{12}\xi\sigma_L\sigma_I(1-2\pi); \end{aligned} \tag{27}$$

$$D_\theta = \kappa(\eta - \theta) + \rho_{23}\sigma_I\nu\sqrt{\theta}(1-\gamma)\pi + \rho_{13}\xi\sigma_L\nu\sqrt{\theta}(1-\gamma); \tag{28}$$

$$D_{\pi\pi} = \frac{1}{2}\sigma_I^2\pi^2(1-\pi)^2 + \frac{1}{2}\xi^2\sigma_L^2\pi^2 - \rho_{12}\xi\sigma_L\sigma_I\pi^2(1-\pi); \tag{29}$$

$$D_{\theta\theta} = \frac{1}{2}\nu^2\theta; \tag{30}$$

$$D_{\pi\theta} = \pi\nu\sqrt{\theta}\left(\rho_{23}\sigma_I(1-\pi) - \rho_{13}\xi\sigma_L\right). \tag{31}$$

In (54), the terms involving ξ can be arranged as follows:

$$h(\xi) = D_\xi \xi + \frac{1}{2}D_{\xi\xi}\xi^2, \tag{32}$$

where

$$
\begin{aligned}
D_\xi &= (\mu_L - r) - \gamma\rho_{12}\sigma_L\sigma_I\pi + \pi[-(\mu_L - r) - \gamma\rho_{12}\sigma_L\sigma_I(1-2\pi)]\phi_\pi \\
&\quad +\rho_{13}\sigma_L\nu\sqrt{\theta}(1-\gamma)\phi_\theta + \pi^2(1-\pi)(-\rho_{12}\sigma_L\sigma_I)(\phi_{\pi\pi} + (1-\gamma)\phi_\pi^2) \\
&\quad +\pi\nu\sqrt{\theta}(-\rho_{13}\sigma_L)(\phi_{\pi\theta} + (1-\gamma)\phi_\pi\phi_\theta)
\end{aligned}
$$

and

$$
D_{\xi\xi} = -\gamma\sigma_L^2 + 2\pi\gamma\sigma_L^2\phi_\pi + \pi^2\sigma_L^2(\phi_{\pi\pi} + (1-\gamma)\phi_\pi^2).
$$

Hence, the optimal allocation to the liquid stock in the Unconstrained Case is as follows:

$$
\xi_t^* = -\frac{D_\xi}{D_{\xi\xi}} \tag{33}
$$

and that in the Constrained Case is as follows:

$$
\xi_t^* = \begin{cases} 1-\pi_t, \ \text{ if } -\frac{D_\xi}{D_{\xi\xi}} \geq 1-\pi_t, \\ -\frac{D_\xi}{D_{\xi\xi}}, \ \text{ if } 0 < -\frac{D_\xi}{D_{\xi\xi}} < 1-\pi_t, \\ 0, \ \text{ if } -\frac{D_\xi}{D_{\xi\xi}} \leq 0, \end{cases} \tag{34}
$$

where the partial derivatives involved in D_ξ and $D_{\xi\xi}$ are evaluated at (π_t, θ_t, t).

Then, similar to Dai and Zhong (2010), we consider the following penalty approximation of equation (17)–(18):

$$
\sup_\xi \mathcal{L}_1^\xi\phi + \phi_t + K\max\{0,\ -\phi_\pi\} + K\max\{0,\ -\theta + (1+\theta\pi)\phi_\pi\} = 0, \tag{35}
$$

$$
\phi(\pi, \theta, T) = \ln(1-\theta\pi^-), \tag{36}
$$

where $K \gg 0$ is a large penalty parameter. Equation (35) and boundary condition (36) are solved together using the policy iteration algorithm combined with a finite-difference scheme. The detailed procedures of the numerical implementation are omitted here but are available from the authors upon request.

B Procedure of the QMLE

We first rewrite equations (3), (4), and (5) in the following way to perform the QMLE method:

$$d\ln S_{Lt} = (\mu_L - \frac{1}{2}\sigma_L^2)dt + \sigma_L dZ_{1t}, \tag{37}$$

$$d\ln S_{It} = (\mu_I + \lambda\theta_t - \frac{1}{2}\sigma_I^2)dt + \sigma_I dZ_{2t}, \tag{38}$$

$$d\theta_t = \kappa(\eta - \theta_t)dt + \nu\sqrt{\theta_t}dZ_{3t}. \tag{39}$$

Next, given a series of time points $t_0 < t_1 < ... < t_N$, we discretize the above system using the following Euler scheme.

$$\ln S_{L,t_{i+1}} = \ln S_{L,t_i} + (\mu_L - \frac{1}{2}\sigma_I^2)\Delta t_i + \sigma_L\sqrt{\Delta t_i}n_{1i}, \tag{40}$$

$$\ln S_{I,t_{i+1}} = \ln S_{I,t_i} + (\mu_I + \lambda\theta_{t_i} - \frac{1}{2}\sigma_I^2)\Delta t_i + \sigma_I\sqrt{\Delta t_i}n_{2i}, \tag{41}$$

$$\theta_{t_{i+1}} = \theta_{t_i} + \kappa(\eta - \theta_{t_i})\Delta t_i + \nu\sqrt{\theta_{t_i}^+}\sqrt{\Delta t_i}n_{3i} \tag{42}$$

where $\Delta t_i = t_{i+1} - t_i$ and (n_{1i}, n_{2i}, n_{3i}) are three standard normal random variables with correlation coefficient matrix Λ defined by (6). Moreover, the sequences $\{n_{1i}\}$, $\{n_{2i}\}$, and $\{n_{3i}\}$ are all serially independent.

Let $x_i = (\ln S_{L,t_i}, \ln S_{I,t_i}, \theta_{t_i})$ be the vector of state variables at time t_i. Then, the transition density from x_i to x_{i+1} is $|J_{t_i}|f(y_{i+1}; 0, \Lambda)$, where

$$J_{t_i} = \frac{1}{\sigma_L\sigma_I\nu\sqrt{\theta_{t_i}^+}(\Delta t_i)^{3/2}} \tag{43}$$

is the Jacobi determinant, $f(x;\mu,\Sigma)$ is the normal three-dimensional density function with mean vector μ and covariance matrix Σ, and

$$y_{i+1} = \begin{bmatrix} \frac{\ln S_{L,t_{i+1}} - \ln S_{L,t_i} - (\mu_L - \frac{1}{2}\sigma_I^2)\Delta t_i}{\sigma_L\sqrt{\Delta t_i}} \\ \frac{\ln S_{I,t_{i+1}} - \ln S_{I,t_i} - (\mu_I + \lambda\theta_{t_i} - \frac{1}{2}\sigma_I^2)\Delta t_i}{\sigma_I\sqrt{\Delta t_i}} \\ \frac{\theta_{t_{i+1}} - \theta_{t_i} - \kappa(\eta - \theta_{t_i})\Delta t_i}{\nu\sqrt{\theta_{t_i}^+}\sqrt{\Delta t_i}} \end{bmatrix}. \tag{44}$$

We ignore the distribution of the initial BAS (i.e., θ_0), as it is asymptotically irrelevant. Then, the log likelihood function of the entire data sample can be expressed as follows:

$$L = \sum_{i=0}^{N-1} \ln |J_{t_i}| + \ln f(y_{i+1}; 0, \Lambda). \tag{45}$$

We then numerically search for the set of model parameters that maximize L.

C Details of the Model in Section 3.5

Solving the Manager's Problem In the presence of relative performance concerns, with a slight abuse of notation, we denote the fund manager's value function as $J(x, y, \theta, S, t)$. Then, it is governed by the following HJB equation.

$$\max \left\{ \sup_{\omega} \mathcal{L}_0^{\omega} J + J_t, \;\; \mathcal{S}_0 J, \;\; \mathcal{B}_0 J \right\} = 0, \tag{46}$$

$$J(x, y, \theta, S, T) = \frac{1}{1-\gamma}((x + y^+ - (1+\theta)y^-)^{1+p} S^{-p})^{1-\gamma}, \tag{47}$$

on an appropriate solution domain, where the differential operators in (46) are given by

$$\begin{aligned}
\mathcal{L}_0^{\omega} J \;=\;& (rx + \omega(\mu_L - r))J_x + (\mu_I + \lambda\theta)yJ_y + \kappa(\eta - \theta)J_\theta + \mu_L S J_S + \frac{1}{2}\omega^2\sigma_L^2 J_{xx} + \frac{1}{2}\sigma_I^2 y^2 J_{yy} \\
& + \frac{1}{2}\nu^2\theta J_{\theta\theta} + \frac{1}{2}\sigma_L^2 S^2 J_{SS} + \rho_{12}\omega\sigma_L\sigma_I y J_{xy} + \rho_{13}\omega\sigma_L\nu\sqrt{\theta} J_{x\theta} + \omega\sigma_L^2 S J_{xS} \\
& + \rho_{23}\sigma_I y \nu\sqrt{\theta} J_{y\theta} + \rho_{12}\sigma_L\sigma_I y S J_{yS} + \rho_{13}\nu\sigma_L\sqrt{\theta} S J_{\theta S},
\end{aligned} \tag{48}$$

$$\mathcal{S}_0 J = J_x - J_y, \tag{49}$$

$$\mathcal{B}_0 J = J_y - (1+\theta)J_x. \tag{50}$$

By exploiting the homogeneity of the problem, we can use the following transformation to simplify the problem.

$$J(x, y, \theta, S, t) = \frac{1}{1-\gamma}(x+y)^{(1+p)(1-\gamma)} S^{-p(1-\gamma)} e^{(1-\gamma)\phi(\pi,\theta,t)}, \tag{51}$$

where $\pi = \frac{y}{x+y}$ and $\phi(\pi, \theta, t)$ is the scaled value function of the fund manager. Let $\xi = \frac{\omega}{x+y}$.

Then, $\phi(\pi,\theta,t)$ solves

$$\max\left\{\sup_{\xi}\mathcal{L}_1^{\xi}(p,\gamma)\phi+\phi_t,\;\; -\phi_\pi,\;\; -\theta(1+p)+(1+\pi\theta)\phi_\pi\right\}=0, \tag{52}$$
$$\phi(\pi,\theta,T)=(1+p)\ln(1-\theta\pi^-), \tag{53}$$

where the differential operator $\mathcal{L}_1^{\xi}(p,\gamma)$ in (52) is

$$\begin{aligned}\mathcal{L}_1^{\xi}(p,\gamma)\phi &= D(p,\gamma)+D_\pi(p,\gamma)\phi_\pi+D_\theta(p,\gamma)\phi_\theta+D_{\pi\pi}(p,\gamma)(\phi_{\pi\pi}+(1-\gamma)\phi_\pi^2)\\ &\quad +D_{\theta\theta}(p,\gamma)(\phi_{\theta\theta}+(1-\gamma)\phi_\theta^2)+D_{\pi\theta}(p,\gamma)(\phi_{\pi\theta}+(1-\gamma)\phi_\pi\phi_\theta),\end{aligned} \tag{54}$$

with the following coefficients:

$$\begin{aligned}
D(p,\gamma) &= [r(1-\pi)+\xi(\mu_L-r)](1+p)+(\mu_I+\lambda\theta)\pi(1+p)-p\mu_L\\
&\quad +\frac{1}{2}\xi^2\sigma_L^2(1+p)((1+p)(1-\gamma)-1)+\frac{1}{2}\pi^2\sigma_I^2(1+p)((1+p)(1-\gamma)-1)\\
&\quad +\frac{1}{2}\sigma_L^2p(p(1-\gamma)+1)+\rho_{12}\xi\sigma_L\sigma_I\pi(1+p)((1+p)(1-\gamma)-1)\\
&\quad -\xi\sigma_L^2p(1-\gamma)(1+p)-\rho_{12}\sigma_L\sigma_I\pi(1-\gamma)p(1+p);\\
D_\pi(p,\gamma) &= -[r(1-\pi)+\xi(\mu_L-r)]\pi+(\mu_I+\lambda\theta)\pi(1-\pi)-\xi^2\sigma_L^2\pi((1+p)(1-\gamma)-1)\\
&\quad +\pi^2\sigma_I^2(1-\pi)((1+p)(1-\gamma)-1)+\rho_{12}\xi\sigma_L\sigma_I\pi(1-2\pi)((1+p)(1-\gamma)-1)\\
&\quad +\xi\sigma_L^2p(1-\gamma)\pi-\rho_{12}\sigma_L\sigma_I\pi p(1-\gamma)(1-\pi);\\
D_\theta(p,\gamma) &= \kappa(\eta-\theta)+\rho_{13}\xi\sigma_L\nu\sqrt{\theta}(1+p)(1-\gamma)+\rho_{23}\sigma_I\pi\nu\sqrt{\theta}(1+p)(1-\gamma)\\
&\quad -\rho_{13}\nu\sigma_L\sqrt{\theta}p(1-\gamma);\\
D_{\pi\pi}(p,\gamma) &= \frac{1}{2}\xi^2\sigma_L^2\pi^2+\frac{1}{2}\sigma_I^2\pi^2(1-\pi)^2-\rho_{12}\xi\sigma_L\sigma_I\pi^2(1-\pi);\\
D_{\theta\theta}(p,\gamma) &= \frac{1}{2}\nu^2\theta;\\
D_{\pi\theta}(p,\gamma) &= -\rho_{13}\xi\sigma_L\nu\sqrt{\theta}\pi+\rho_{23}\sigma_I\pi\nu\sqrt{\theta}(1-\pi).
\end{aligned}$$

The model is solved in a similar manner as the baseline model.

Solving for the Fund Investor's Value Function Next, we explain how we obtain the fund investor's value function, given the manager's optimal trading strategies. First, we define the following regions:

$$BR \equiv \{(\pi, \theta, t) : -\theta(1+p) + (1+\theta\pi)\phi_\pi = 0\}; \tag{55}$$

$$SR \equiv \{(\pi, \theta, t) : -\phi_\pi = 0\}; \tag{56}$$

and

$$NTR \equiv \{(\pi, \theta, t) : 0 < \phi_\pi < \frac{\theta(1+p)}{1+\theta\pi}\}. \tag{57}$$

We also define

$$\xi^*(\pi, \theta, t) = \arg\max_{\xi} \mathcal{L}_1^{\xi}(p, \gamma)\phi. \tag{58}$$

Then, we denote the investor's value function as $J^i(x, y, \theta, t)$, which also takes the following functional form

$$J^i(x, y, \theta, t) = \frac{1}{1-\gamma_I}(x+y)^{1-\gamma_I} e^{(1-\gamma_I)\phi^i(\pi,\theta,t)}. \tag{59}$$

It can be verified that the function $\phi^i(\pi, \theta, t)$ satisfies the following PDE:

$$\mathcal{L}_1^{\xi^*(\pi,\theta,t)}(0, \gamma_I)\phi^i + \phi^i_t = 0, \text{ if } (\pi, \theta, t) \in NTR, \tag{60}$$

$$-\phi^i_\pi = 0, \text{ if } (\pi, \theta, t) \in SR, \tag{61}$$

$$-\theta + (1+\pi\theta)\phi^i_\pi = 0, \text{ if } (\pi, \theta, t) \in BR \tag{62}$$

with the terminal condition:

$$\phi^i(\pi, \theta, T) = \ln(1 - \theta\pi^-). \tag{63}$$

After we obtain the scaled value function $\phi^i(\pi, \theta, t)$, the investor's annualized certainty equivalent rate of return, measured at the initial time $t = 0$, is simply $\phi^i(\pi_0, \theta_0, 0)/T$.